Linear Programming: Fundamentals and Applications

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To

DR. H. B. MAYNARD

Preface

This book is intended primarily but not exclusively for people engaged in management activities at all levels in the firm and students of management wherever they may be. Its purpose is to point the way to the increased use of linear programming for solving certain types of management problems.

Briefly, linear programming is a technique or tool for providing management with information on which to base decisions and planning. It is a practical technique that, on the basis of results to date, is capable of providing management with better information about certain problems than many techniques and methods currently in use.

Linear programming differs from the "usual" planning procedures found in business in at least two ways. First, the user can consider, relate, and weigh *more* information and facts about a problem or situation than ordinary pencil-and-paper procedures allow, and in doing so he can gain a clearer picture of the business as a whole, as well as the operation of his own particular area. This is especially useful in large, complex problems where the proper relationships tend to be obscure. Second, an optimum or best answer is provided that the user can verify and support. These features—clearer relationships and supportable best answers—are distinguishing features which are of value to management in making better use of the firm's resources.

We believe that there is considerable use for linear programming in industry. We believe further that the student at the university and college level stands to gain much from the study of linear programming because of the valuable and useful insight into business and management problems that he will obtain as a result.

In this book we will emphasize and stress use and application rather than mathematics and theory. The developers—mathematicians, economists, and researchers—have provided a valuable tool. The concepts, principles, and mathematics underlying the procedure have been proved valid and have appeared in the texts and articles mentioned in the bibliography. The job to be done now is to show how this tool can be used

to increase insight into management problems and to improve managerial and firm performance. Our objective in writing this book is *to show how that can be done.*

We propose to accomplish the objective by translating the principles and concepts into practical terms by using familiar industrial examples. Then by using actual industrial problems we hope to demonstrate the usefulness and value of the technique to management and students.

The material has been divided into separate sections for ease of reading. The sections are "Introduction," "Methods," "Application," and "Technical Appendixes." Each section is self-contained so that the reader can select only those sections of particular interest. For example, certain management personnel not interested in the mechanics of the methods can bypass "Methods" and concentrate on the "Introduction" and "Application." Those who are interested solely in the methods can concentrate on "Methods" to the exclusion of the other sections.

Section I, "Introduction," introduces linear programming generally from a management point of view. It discusses the origin, need for, and value of linear programming and cites a number of the problems to which it has been applied. In general, the section attempts to state why management should be interested in linear programming.

Section II, "Methods," presents the technical aspects of the methods. This section has been written for the individual who has to understand linear-programming methods in order to set up the problem and see that the calculations are carried out. The principles, concepts, steps, and procedures are developed through the use of industrial problems.

Section III, "Application," presents application techniques and experiences. Suggestions are given for recognizing linear-programming problems and developing industrial applications by the use of a model. Several timely problems are worked out in complete detail to assist the reader with his own application.

Section IV, "Technical Appendixes," contains the mathematical discussion of the simplex method for those interested in the basic mathematics underlying the technique and methods.

ACKNOWLEDGMENTS

A book of this kind is never completely the work of one or two people. It evolves from experience, exchange of ideas, and association with others. We therefore want to take this opportunity to acknowledge the contributions to and influences of others on our work.

We first want to express our appreciation to Dr. H. B. Maynard, president of the Methods Engineering Council, for providing the opportunity

to learn, develop, and use linear programming. His willingness to underwrite research and development has made this book possible.

Our introduction and early training in the field were obtained from Professor W. W. Cooper, of the Carnegie Institute of Technology, and from Professor A. Charnes, of Purdue University. In addition, their assistance was an important factor in the success of the first applications and an influence on later developments.

We also want to acknowledge the contributions, comments, and criticisms of Mr. Donald W. Moffett, senior consultant of the Methods Engineering Council, which were most helpful in the preparation of the book.

In addition to the acknowledgments of technical assistance, we want to express appreciation to Mr. R. H. DeMott, chairman of the board of SKF Industries, Inc., for our first opportunity to apply linear programming to a business problem.

Our thanks also go to the Principal Group of H. B. Maynard and Company, Inc., for carrying the added load of work while this book was being written.

Last but not least, we wish to thank our associates, especially those who, working with us on numerous assignments, have contributed to the knowledge and experience expressed in these pages, and to Mrs. Jeanne Bauman for the task of typing and editing the manuscript.

It is hoped that this book will serve two purposes: (1) to provide a summary of ideas and applications that may be useful to others, and (2) to advance the development and expansion of the use of what we believe to be a most important management tool.

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SECTION ONE

Introduction

Linear Programming is a new and somewhat strange term to business. The name Linear Programming does not tell very much about what this technique can do or the results that it can bring. Consequently, if it is to gain wider acceptance by those who can profitably use it in industry, it needs to be defined and explained in understandable terms.

The purpose of this introductory section, therefore, is to acquaint business people with linear programming from a management point of view and to stress some of its special features and results where it has been used. In the process, it will become apparent that linear programming is not as strange and unfamiliar as it might appear at first glance. The linear-programming objective—the most effective use of firm resources—is a familiar and desirable one, and calculating the *most effective* course of action that makes the objective possible is another feature of linear programming that will interest executives and managers at all levels of the firm.

This first section provides general information and an introduction to the how, what, why, and where of linear programming for management personnel. Further, it describes some of the types of problems linear programming has solved. In general, it gives a number of answers to the question: "Why should I be interested in linear programming?" The answer, based on results, is: "Linear programming provides a better insight into many basic problems of management and assistance and information for solving them more effectively than is obtained with the usual paper and pencil methods."

CHAPTER 1

Linear Programming—an Aid to More Effective Management Decisions

In recent years there has been a considerable increase in the recognition of the importance of factual information as a basis for making better decisions at all levels of management. As a result more and more executive time and effort are devoted to determining what facts and information are needed, directing their gathering and processing, and then interpreting the resulting information into a usable program. Many times reliable facts are difficult to obtain and frequently more difficult to interpret, to say nothing of the problem of deciding what facts are needed in the first place.

Businessmen and students of business problems have been wrestling with the problem of "facts" for many years. Many men have become famous and wealthy by being able to make reliable decisions when there were few facts available. Many other men have devoted long periods of their life to finding better ways of handling facts and business information.

Linear programming is a significant advance in the techniques for handling facts and providing information. It was developed by economists and mathematicians for the purpose of determining how to use limited amounts of resources to best advantage. This problem is a most familiar one to the businessman who is constantly confronted with the necessity of forgoing the opportunity of making one move in order to be able to make another move. This forgoing of one opportunity in order to take advantage of another usually involves making some rather difficult decisions. The real difficulty in making such decisions usually lies in the areas of selecting, gathering, processing and interpreting facts and information. Further, in many cases the businessman has approximations and indefinite statements rather than explicit facts when he is selecting the places to use his limited resources. This condition causes decision making to be even more complex and difficult than otherwise.

An ideal decision-making situation for the businessman should have at least two basic types of information present: First, the businessman

should know how much man power, capital, floor space, equipment, and inventory are available. In other words, he should have an accurate measure of his productive resources in terms of demand.

Second, he should know how much profit or measurable benefit he could obtain from using a unit of these resources for each available opportunity. Existing conditions usually are far from these ideal conditions. The businessman does not have many of the facts about either his resources or opportunities, or, if he does, he may not have the means to evaluate them. The information he does have may be so voluminous or so complex that he has difficulty in seeing how to organize and interpret the information into a program that will ensure his obtaining the potential that is possible in his situation.

Although LP was developed to help solve problems of using scarce resources as effectively as possible, it helps to solve the data and information problem of the businessman as well. In doing so, it provides a way of:

1. Organizing and interpreting a large volume of data and facts.

2. Using approximations in a mathematical processing of the data and facts.

3. Evaluating the benefits or profits that should come from apportioning or allocating the available resources to different opportunities or combinations of opportunities. Within certain limits linear programming provides a better way of determining how to assign or use limited resources among different opportunities. Thus, linear programming provides a better way of handling the information used in making decisions.

At this point then we are in a position to define linear programming in management terms on the basis of the information it provides. Our definition is as follows:

> Linear Programming (LP) is a technique for specifying how to use limited resources or capacities of a business to obtain a particular objective, such as least cost, highest margin, or least time, when those resources have alternate uses It is a technique that systematizes for certain conditions the process of selecting the most desirable course of action from a number of available courses of action, thereby giving management information for making a more effective decision about the resources under its control.

It is not important for the executive to know why or how the technique works, although it is important for someone in the firm to have this information. It is important for the executive to realize that the technique does identify a best solution and program under specified conditions that he can use in making decisions.

LINEAR PROGRAMMING AS A TECHNIQUE

Unless you have seen what linear programming can do or know how it is used, the use of the word *technique* may not convey much meaning. To clarify the meaning of the word *technique,* therefore, let us examine a different but analogous system—one that is more familiar.

One of the ways of accounting for costs is by the standard-cost technique. When the term *standard-cost technique* or *system* is mentioned, most people in business know in general what is meant. Because the term is familiar, they are able to conjure a mental picture of what is involved, and in general they are able to follow the discussion. For example, they know that certain numbers and values taken from time cards, production records, receiving reports, various schedules of expense, and the like are manipulated according to certain rules. When the computations have been carried out successfully, the original mass of figures has been reduced to a few summary figures that appear on the accounting statements for management's consideration and planning.

Throughout the entire computational process there is nothing tangible, nothing you can put your hands on. Yet the words *standard-cost system* have meaning and are at least generally understood when said or written by a great many people in business. Linear programming, in a broad sense, is much the same kind of procedure. It, too, starts with information expressed as numbers and, through a series of steps and computational rules, reduces original information to a few summary figures for management's consideration and planning.

Strictly speaking, linear programming is a mathematical technique. In its original concept it was based on principles that require a knowledge of advanced mathematics In actual practice, however, linear programming has been simplified to a point where its routine application requires little more than an understanding of high school algebra. A competent industrial engineer or cost analyst should not have too much difficulty in learning how to set up and solve many linear-programming problems. He might, however, find it more difficult to define the problem and select the data that are to be processed. Usually a reasonable amount of instruction and guided experience, plus the exercise of some ingenuity, will enable such men to handle successfully many linear-programming problems, thereby making available useful information for the decision maker to use.

As a computational technique, linear programming draws upon mathematics that are not familiar to business people. The mathematics, some of which is new in concept, makes it possible to process facts and data more effectively than many pencil-and-paper methods, thereby provid-

ing better information. A comparison of high school algebra with the new concepts will perhaps point out why LP is different and at the same time will provide a better insight into the technique itself.

Most of us learned in high school how to solve the following problem by using algebra:

> What is the price per ton each of coal and limestone when 4 tons of coal plus 6 tons of limestone cost $32 and 5 tons of coal plus 5 tons of limestone cost $35?

By the use of simultaneous equations the solution to the problem can be calculated as follows:

Cost of 4 tons of coal + 6 tons of limestone = $32 (1-1)
Cost of 5 tons of coal + 5 tons of limestone = $35 (1-2)

Multiplying Equation (1-1) by 5 and Equation (1-2) by 4 gives the following result:

Cost of 20 tons of coal + 30 tons of limestone = $160 (1-3)
Cost of 20 tons of coal + 20 tons of limestone = $140 (1-4)

Subtracting Equation (1-4) from Equation (1-3), we obtain the following price for each ton of limestone:

Cost of 10 tons of limestone = $20
Cost of 1 ton of limestone = $ 2 (1-5)

Substituting this value in either Equation (1-1) or Equation (1-2), we obtain the following price for each ton of coal:

Cost of 4 tons of coal + 6 ($2) = $32
Cost of 4 tons of coal = $20
Cost of 1 ton of coal = $ 5 (1-6)

The solution of sets of simultaneous equations of this kind is fairly straightforward and has been known for a long time. Unfortunately, simultaneous or algebraic methods for solving equations are not practical for solving the large problems generally found in industry for a number of reasons. For one thing, attempting to solve industrial problems by the method of determinants or simultaneous equations becomes too involved and time-consuming. For another thing, it may not be possible to formulate as many equations as there are unknowns, which is a necessary requirement for simultaneous solution. In addition, there is nothing inherent in the simultaneous or algebraic methods that ensures a *best* solution when the calculations are completed. Finally, it is not possible to express most business problems as nice, neat equations. For example,

we are more likely to find the problem of coal and limestone to be along the following lines:

> The cost of 4 tons of coal plus the cost of 6 tons of limestone will be *at least $32;* the cost of 5 tons of coal plus the cost of 5 tons of limestone will be *at least $35;* there is $135 available for coal and limestone during the period, and we must purchase *at least* 2 tons of limestone. Find the largest number of tons of coal and limestone that can be purchased.

Under these conditions we do not know how much the actual cost of the coal and limestone will be. Further, we do not know whether or not all of the available money will be used in the purchase. For example, there may be up to $1.99 remaining after the transaction is completed. The indefiniteness of the latter problem has moved it out of the realm of simple algebra. This indefiniteness, however, is characteristic of many business problems. Another characteristic of business problems is the large amount of data usually involved—which may or may not be well organized. The combined effect of these two data and informational characteristics is that the businessman frequently does not use the available information to the best advantage. This failure to take full advantage of the available information is not due to his knowledge that the information contained estimates and approximations. Nor is it due to the fact that more information is frequently needed. Rather, the failure to use fully the available information is due to lack of a technique for organizing, processing, and interpreting the information. Linear programming provides a way of using data and facts to arrive at some decisions that were formerly arrived at on the basis of intuitive judgment.

As we shall see presently, the use of linear programming involves much more than the knowledge of technique. It requires and develops a certain approach to and a way of looking at management problems. The discussion of linear programming as a technique, however, serves to introduce us to this new management tool.

ORIGIN AND BACKGROUND OF LINEAR PROGRAMMING

The origin and background of linear programming has been in the field of econometrics and research into economic relationships. Its roots go back to 1874 and the works of the mathematical economist Leon Walras. In his work *Eléments d'Economie Politique* ("Elements of Political Economy") Walras showed that the price of any number of commodities at a single time can be determined by solving simultaneously the correct number of equations in terms of the number of unknowns for which a solution is sought. At the time, the concept was

revolutionary, and today it is recognized as a contribution to economics. It was this first attempt to solve problems of scarcity by stating problem conditions in equation form that provides the connection between Walras and linear programming. The method of solving the equations for solution by LP is completely different from that used by Walras, however.

Linear programming, as it is known today, is descendant from the input-output method of analysis developed by the economist Wassily W. Leontief in the 1920s. The present-day development stems primarily from the work of George B. Dantzig, who is credited with developing the Simplex Method of linear programming, which is essentially a method of solving simultaneous equations and inequations for an optimum or best solution. The adaptation and application of the simplex method to solving industrial problems has come about in just the last few years, despite the fact that Dantzig announced his development in 1947. Today, linear programming is considered by many to be a technique or tool of a wider field of analysis called Operations Research.

OPERATIONS RESEARCH AND LINEAR PROGRAMMING

Practically speaking, Operations Research, or OR, as it is popularly known, is the scientific approach to and analysis of business problems aimed at providing executives and management with information for most effective operation of the business. Within this concept, linear programming is a technique or tool of operations research.

The beginnings of OR as we know it today took place during World War II in Britain. Groups of scientists were assembled, as a last-ditch measure or desperation move, to provide assistance with several critical war problems. The two most notable of these were the German air blitz and the German U-boat menace. The assistance rendered by the scientists in solving these problems was invaluable. It is generally conceded that their work was a successful and substantial contribution to the war effort.

As a result of their efforts, it became apparent that improvement in the *use of existing weapons and equipment* could contribute more in a short time than could improvement of the weapons or equipment themselves. This is an important consideration from management's point of view, in business especially, where immediate problems are pressing.

The scientists assembled to do OR work used the tools with which they were most familiar—mathematics and analytical and logical methods developed in their own specialist fields. The results obtained with these tools were as surprising as they were gratifying. The concept that specialists in physics, chemistry, mathematics, or any other scientific field could operate in combination to produce effective answers to problems

in areas totally different from the experience of any of them was startling and difficult to accept at the time. This was especially true because, taken individually or separately, it was unlikely that any one of the scientists could have solved the problem himself; but as a group they were particularly effective.

These successes soon led to the extension of operations-research activities to the military organization of the United States. Since then many military problems have been solved and are continuing to be solved by OR teams. OR groups are now an integral and accepted part of military organizations, and their number, although small in 1957, is increasing in industry.

SOME IMPORTANT FUNDAMENTALS FOR UNDERSTANDING AND USING LP

If you were to ask a mathematician to define linear programming you would probably be told that it is a method of maximizing or minimizing a linear function subject to linear restrictions and constraints.

If you were to ask an economist the same question, you would probably hear something to the effect that linear programming is a technique for allocating a group of limited resources among a number of competing demands where all decisions are interlocking because they are made under a common set of fixed limits.

As we have seen, linear programming, in the industrial sense, is a mathematical technique for allocating or using limited resources in such a way as to make best use of them in terms of a predetermined objective, such as least cost or highest profit margin.

Actually, all three definitions say the same thing in different words and when considered together help to highlight some of the underlying concepts as well as to indicate the different ways in which different people in business view the same technique.

1. Some underlying concepts

One of the basic concepts of linear programming is the concept of an *activity*. An *activity* in the business sense is a specific method for carrying out a task or making a product. For example, the production of a Ford wrap-around windshield on a specific production line at Plant A is an activity. The production of the identical windshield in a different plant having a different rate and cost of production is a different activity.

Another basic concept of linear programming is the concept of the *alternative*. There must be alternatives of one kind or another, such as machines or processes, that can be employed to accomplish the desired objective (the products must be able to compete for use of the machines or processes). Then, the decision about which machines or processes

are to be used, and the extent to which each is to be employed in obtaining the objective, is calculated by linear programming. LP determines, for the conditions of the problem, the optimum levels at which the machines, space, or processes are to be used in order to obtain the predetermined objective.

Still another concept, helpful in understanding linear programming, is the concept of *linearity*, or *being linear*. For our purposes, *linearity* means that a change in the number of departmental productive hours will bring a proportional change in output and profit margin. If we attempted to plot the relationships involved on rectangular coordinate axes, they would appear as straight lines somewhat similar to the lines that appear on the break-even chart.

Programming means calculating for a fixed time period and set of conditions the solution to a set of linear equations and inequations.

2. Requirements for setting up a problem for solution

Certain requirements, over and above computational requirements, need to be met to process data and facts by LP methods. The first requirement is to state the problem in terms of an objective. Typical examples are as follows:

> Determine the combination of products which can be produced with the available resources that will earn the largest manufacturing margin.
>
> Find the least-cost program for making a list of parts on the available equipment.
>
> Establish the most profitable division of available capital among new equipment, product development, and advertising.

It is important that the problem be stated as explicitly as possible.

The next requirement is to determine the information that is to be used in solving the problem. When LP is used it is generally possible to include any information that may have an effect on the answer to the problem. The computational routines will usually ignore any information that does not affect the answer to the problem.

The third requirement is to gather the needed information. It must be possible to state the data and facts of the problem in terms of units of measure. These need not be specific, but they must be stated in terms of numbers. The statement may simply be that some quantity will not be more than (or less than) a stated value. For example, "The sales of the De Luxe Model in the Buffalo territory will not exceed 30 units" is such a statement. It must be possible also to express the profits or benefits that are expected from the use of resources in specific terms. "Dollars

profit per unit," "hours saved per 100 pieces," or "freight costs per piece" are typical examples.

The fourth requirement is to put the information into the form of related linear equations and inequations. Equations are specific statements in mathematical form. For example, the statement that the cost of 4 tons of coal plus 6 tons of limestone totals $32 expressed as an equation is $4C + 6L = 32$. Inequations are approximations in mathematical form. For example, the statement that the cost of 4 tons of coal plus 6 tons of limestone will be at most $32 expressed as an approximation or inequation is $4C + 6L \leqq 32$.

When all these conditions are met, linear programming can generally be used to organize, process, and interpret the available data and facts into usable information. Further, linear-programming answers can be used to determine how to use the available resources most effectively.

It frequently happens that either by choice or by necessity the decision involves selecting a combination of several opportunities from many opportunities and combinations thereof. LP provides a way of determining the combination of opportunities that will use the resources to the best advantage. The selected combination of opportunities becomes a program for the utilization of the available resources.

SOME MANAGEMENT AREAS IN WHICH LP HAS BEEN USED

The following list is not a complete list of LP applications by any means. It is, however, an indication of the problems that have been solved and the results that have been obtained up to this time.

1. Determining the most profitable product mix to be obtained from existing facilities
2. Determining which parts to make and which to buy to obtain maximum profit margin
3. Scheduling orders to machine centers at least cost consistent with delivery promises
4. Establishing the best location of warehouses to minimize transportation costs
5. Selecting equipment and evaluating methods improvements that maximize profit margin
6. Planning profits on a fiscal-year basis to maximize profit margin from net investment in plant facilities and equipment, cash on hand, and inventory.
7. Supplying a fluctuating sales demand at least inventory cost considering a fixed level of production and stabilized employment
8. Allocating production releases among several plants so as to maximize profits considering manufacturing and distribution costs

9. Determining equitable salaries and sales-incentive compensation

10. Determining the feed mix that satisfies nutritional requirements and minimizes the cost of raising livestock

11. Programming a chemical-distilling-type operation, including the processing of purchased material to obtain highest manufacturing margin within the limits of sales demand

12. Planning the most profitable match of sales requirements to plant capacity that obtains a fair share of the market

SOME ADVANTAGES TO MANAGEMENT FROM USING LP

Now that we have defined LP in a number of ways, examined its origin and background, considered some of the basic concepts, and seen a number of problems to which it has been applied successfully, we are now ready to examine its value and use critically in terms of direct benefits to the individual and the firm.

Experience to date indicates that much of the importance and value of LP arises from its contribution to increasing the effectiveness of the individual executive and manager, accelerating the development of subordinates, and improving the operation and profit position of the business. Each of these contributions will be discussed in turn.

1. Executive and managerial effectiveness

A knowledge of linear-programming techniques improves the effectiveness of the individual executive and manager in several ways:

1. *Insight and perspective into problem situations.* An executive, manager, or any user is compelled to organize the facts and information about a problem before LP methods can be applied properly. In doing so, a better perspective and a keener insight into the problem frequently results. Actually, LP forces logical organization and study of information in the same way that the scientific approach to a problem requires. This generally results in a clearer picture of the true problem, which frequently is as valuable and revealing as the answer itself because it leads more surely to dealing with *causes* rather than *effects*—solutions rather than stop-gap expedients.

The accumulation and comparison of quantified conditions of a problem can strip away much false impression and fallacious reasoning and permit accurate and realistic appraisal.

2. *Consideration of all possible solutions to problems.* Many management problems are so complex that difficulty is encountered in planning *any* feasible solution, let alone an optimum solution. By using linear programming, the manager makes sure he considers the best solution or solutions as well as any others that he might want to consider.

3. *Better and more successful decisions.* With linear programming, the executive or manager builds into his planning a true reflection of the limitations and restrictions under which he must operate. He learns the true effect of the many variables he must consider. Furthermore, when it is necessary to deviate from the best program, the manager can evaluate the cost or penalty involved. He will then have an idea of what he forgoes if he elects to follow a course of action other than that indicated by LP.

4. *Guidance in giving specific orders.* A linear-programming solution comes out in specific quantitative terms:

How many shall we make?
How shall we make them?
Where shall we make them?
How many shall we carry in inventory?
What are the costs, profits, and the like?

With this type of information, the manager knows what specific orders and instructions to issue to get the results that are possible. Furthermore, the LP answer provides a calculated yardstick against which actual performance can be measured and controlled.

5. *Better tools for adjusting plans to meet changing conditions.* Once a basic plan is arrived at through linear programming, the basic plan can be reevaluated for changing conditions. Plans can be laid for several sets of conditions to find out how to prepare best for possible future changes. If conditions change when the plan is partly carried out, changes can be determined so as to adjust the remainder of the plan for best results.

2. Subordinate development

Perhaps one of the most important and overlooked aspects of the new decision-making techniques, including linear programming, is their influence on the development and training of management personnel. In 1957 many executives are wrestling with the problem of how to prepare junior executives and middle management men for larger responsibilities, to say nothing of how to improve the effectiveness of management personnel in present positions, especially when talent is in such short supply.

Training and experience in applying LP will help with this problem. Today a number of authorities in the field believe that the real power of the OR techniques lies not so much in their more effective and profitable solutions to some long-standing management problems as in their ability to provide management people with an increasingly clear vision of the business as a whole or as a system. This viewpoint provides a better basis for an understanding by all managers of their own re-

sponsibilities inside and outside the divisions or departments they run.

Another plus factor in the development of people concerns experience. You have heard the remark made many times that there is no substitute for experience—experience that has taken years to accumulate—as a guide to management thinking, action, and decision making. There is a great deal of truth to the statement, provided the firm can afford to wait for the individual to obtain the experience and the experience has been of sufficient quality to have value, once acquired. Fortunately, LP makes it possible to shorten the experience and learning time required for successful performance in a particular position. The insight into problems and relationships provides know-how and perspective that might otherwise take years to develop in a person on the basis of experience alone.

Finally, LP and the other OR tools where they apply help to provide confidence that will encourage subordinates to make decisions. Because LP is a computational system and because the technique provides best answers and variations to best answers, subordinates have a tool and information that make deciding easier and permit greater delegation.

3. Business operation

It is natural to expect that, as the effectiveness of individual executives is increased, as the pace of development and training is accelerated, and as direct applications are made, improvement in the operation of the business will result. And such has been the case.

The growing number of case histories that are appearing in management literature is proof of this. The next section, containing actual cases, spells out in considerable detail a number of actual problems solved through the application of LP. Discussion of the dollars-and-cents answers and other values to management are also covered. If we disregard for a moment the human factors, we find that improvement comes in several ways. One source of improvement in operation of the business stems from the fact that LP can indicate how to make most effective use of *existing facilities.* The word "existing" is emphasized because frequently improvement is obtained without the addition of equipment and people—merely by managing more effectively the facilities and resources that are available.

Then, after those possibilities are explored, it is possible to plan for more profitable operation through the addition or purchase of new facilities and equipment by considering such possibilities in the calculations. In this way the profit-making potential of new equipment can be computed in advance—before commitment to purchase is made.

Perhaps the best way to summarize the value and use of linear programming and keep it in proper perspective is to quote part of an address

made by Robert E. Lewis, president of Argus Cameras, Ann Arbor, Michigan.[1]

> Reviewing what we believe have been our accomplishments in linear programming . . . we feel specifically that we have reduced costs and increased control. In a more general sense, we are now making decisions objectively rather than flying by the seat of our pants. We also have more confidence in our ability to remain competitive because we know more about what we're doing.
>
> I think it might be appropriate to say at this point that the confidence engendered by use of these techniques, enables the management to use greater vision in planning for the growth of the company. Decisions that historically have been made by management can now be made at a lower level. This gives our top people more time to devote to the many other problems that constantly present themselves. It is difficult to try to think in terms of growing unless you already have your present efforts under control. . . .
>
> Our embracing of these techniques has substantially aided our communications. Our top management team joins the next lower level in advancing new techniques. The top management committee learns the trend of thinking of the lower level groups and is able to steer the thinking toward the solution of problems. In other words, the use of these techniques provides an excellent transfer of the philosophy of top management.

ACTUAL CASES OF SPECIFIC VALUES WHICH HAVE BEEN OBTAINED THROUGH LP

Linear programming as a management-planning technique offers many advantages. In almost every application made to date, LP produced a plan that would accomplish management's objective better than the usual methods. In some cases, this advantage alone justified training personnel and adopting the technique for the specific problem.

The total benefits derived from the applications, however, were far greater. In those companies which have applied linear programming, it has been demonstrated that the application has produced or led to the following advantages.

1. Profits, costs, and service

One company believed they were faced with serious overloading in their automatic-screw-machine department. Their orders were behind

schedule, and they were planning to install additional machine capacity to take up the overload. Then they applied linear programming to determine whether they were doing the best they could with their present machines. Loading machine centers with an LP method resulted in a sufficient reduction in machine hours to produce the same number of parts, on schedule, on the same machines. Not only was it possible to cancel the order for additional machines, but the total reduction in machine hours amounted to a $12,000 saving per month. Under the new scheduling method, work-in-process inventory is smaller, delivery promises are being met, and the capacity of the shop is such that a considerable increase in volume can be handled without overloading.

2. Executive and managerial planning time

Once a problem has been defined and set up for solution, the computational procedures can be delegated to clerical personnel or programmed on computers.

One company had to allocate production orders to various facilities located at two different plants. This required the constant attention of a top-level executive who had firsthand knowledge of all the operations. To enable him to exercise good judgment, it was necessary for him to spend many hours gathering, studying, and evaluating data on the production capacity, costs, and performance of the different production units.

By using linear programming to allocate production orders, this company makes it possible for the collection and analysis of data to be done by clerks. The results of the linear-programming calculations can then be presented to the responsible executive in the form of an optimum plan, together with analyses of alternative plans and sound, factual data on the cost of deviating from the optimum plan. As a result, much of this executive's time is freed for consideration of long-range-planning problems.

3. Management control information

One management group found that planning production on the basis of a set of policies and rules of thumb provided results almost as good as the optimum program calculated by linear programming (these policies and rules of thumb, incidentally, were developed as a result of using linear programming to study the problem).

In this situation, the answers calculated by linear programming are not used for regular week-to-week scheduling. Instead, the general policies and rules of thumb guide production planning, and the plans are checked occasionally by linear-programming calculations to be sure that the approximations are still valid.

4. New information

Problems which are very large and complex and which involve many different combinations of factors are almost impossible to analyze without a mathematical method such as linear programming. For example, one company produced a product line on several different production facilities at each of two plants. The products were shipped to customers in many locations.

Production costs varied between production facilities. Freight costs depended upon which plant an order was shipped from. Considerations of profitability were further complicated by the fact that one plant consistently ran at a 5 per cent larger variance from standard cost than the other. With this large number of factors to take into account, it is not surprising that production schedules arrived at on the basis of experience and judgment did not consider the effects of the various factors in their proper proportions.

When linear programming was applied to this problem, it was possible to show that undue weight was being given to freight costs in planning the production program. In addition, it was possible to show that a special agreement on pricing and freight absorption which had been reached with one customer was extremely unprofitable to the supplier and failed to provide large savings to the customer.

5. Information for planning future action

When planning for changes which involve investments—for example, the acquisition of new machine capacity—it is essential that management be aware of what the new equipment will do. It is also essential that comparisons of equipment are made on the basis of effective utilization of both the old and the new equipment.

In the application mentioned earlier, the company was prepared to add additional screw-machine capacity because they thought they were doing the best they could with the existing machinery. As was mentioned, linear programming showed that this reasoning was unsound.

Another company, in a similar situation, was making up for lack of capacity by purchasing certain items. When linear programming was applied to the planning of a maximum-profit production schedule, this company found that they could operate certain machine sizes much more profitably than others. On the basis of this factual, quantified information, they were able to plan the disposition of some machines of one size and their replacement with machines of another size and show definitely that this exchange would result in savings.

6. Scheduling previously unscheduled operations

In at least one instance, in a chemical-processing industry, linear programming has made it possible to plan how much time a process should run under certain sets of conditions. In this case, the process produced a different combination of products under each possible set of operating conditions. The linear-programming solution considered the profitability, the demand, and the waste of each one of these products to arrive at a most profitable program. This was in contrast to the earlier scheduling method, which consisted of running that condition which produced the most of the products for which there was the largest demand.

These advantages actually have been obtained by the companies who have applied linear programming. They are in addition to substantial savings made in each case as a result of the optimizing procedure itself. Applications similar to those which developed these advantages for the companies who used them are explained in detail in Section Three, "Application."

SUMMARY

Primarily, the value of linear programming rests in providing executives and managers with information—better, quicker, more factual, and more accurate—so that the executive or manager is in a better position to make decisions about the operations under his control than he would without such information.

It is important to make clear at this point that linear programming does not change the basic character of the decision making that executives and managers have to do. Rather, it permits a clearer definition of the problems that he faces, facilitates the determination and selection of as good a solution as is possible for him to consider, and adds explicitness and certainty to the terms in which he makes his decisions. It will still be necessary for the executive to make the decisions—tools and techniques will not make them for him The tools will provide him with quantitative information to which he can add his assessment of such qualitative information as morale and philosophy of customer service in arriving at a solution to his problem.

It is also apparent to date that the use of the new decision-making tools of which LP is one provides the executive with the following:

1. More time to manage, by providing him with information quicker and permitting more delegation

2. Less pressure to get things done, because he can do more accurate planning and avoid pitfalls that create crisis situations

3. Better-qualified personnel, because the techniques provide insight into management problems and decision making that formerly would have required years of experience and practice to obtain

Each and all of these factors will add to the ability of the manager to manage better, which, in turn, will reflect in increased profitability for the firm.

SECTION TWO

Methods

The concepts, principles, and computational steps of a number of linear-programming methods are described in this section. The basic mathematical theory and derivation is contained in Section Four, "Technical Appendix."

Five methods are explained in detail, and industrial problems are used to demonstrate the important points and concepts of each method. The methods that will be discussed in order of presentation are the Transportation, Modi, Simplex, Ratio Analysis, and Index. Actually all methods are derived or have evolved from the simplex method, which is considered to be the basic linear-programming method.

Despite their common origin, each method requires its own particular arrangement of problem data so that its particular computational process may be used for solving a given problem. The word method, therefore, refers to a particular arrangement of problem information and data, together with the special computational steps used in solving the problem under that arrangement. When we refer to five methods, we mean that there are five different arrangements of problem information and five different computational processes for solving linear-programming-type problems. For example, the use of the simplex method requires a certain arrangement of data to which the special computational process of the simplex method is applied in calculating the solution to the problem. The use of the modi method, on the other hand, demands a different arrangement of problem information, expressed in different form and units, and involves its own particular computational process for solving the problem. The nature of the problem itself generally establishes the computational method to be used. The more complex problems usually require solution by the simplex method. The specialized and less complex types of problems are more likely to be solved by the other more specialized and less time-consuming methods, such as the transportation and index methods.

In Section One, LP was defined as a technique for allocating or using limited resources (such as plant capacity, storage space, and material)

to achieve a specific objective (such as least cost, highest profit margin and greatest quantity), where the limited resources have alternate uses and where a change in the amount of resources brings a corresponding or proportional change in result.

In general, limited resources indicate to the executive what he "cannot do"—the point beyond which he cannot go. Under these circumstances, then, the executive wants to know what "can be done" to use most effectively the resources that he does have. Each of the five methods through its computational process provides information for converting the "cannot do" restrictions and limitations into a "can do" program. Each method, where it applies, permits calculation of the most desirable "can do" program within the "cannot do" restrictions according to profit, cost, or other measure of desirability.

Reduced to essentials, each method involves a formalized trial-and-error process in arriving at a best answer. The general process for solving a problem is to start with a first solution, which evolves from the arrangement of problem information, and calculate improvement in incremental steps—one step at a time—until the method indicates that there is no further improvement to be gained. The formalized trial-and-error part of the process occurs at each incremental step when a decision has to be made regarding the next increment, or step. Of course, each method provides an indicator or guide regarding the next step, but a selection has to be made before the process can continue. Each method indicates through its computational process when the best answer has been determined and that consideration of additional increments or further calculations will result in either a less desirable or at best an equally desirable answer to the one already computed.

The basic steps involved in solving a linear-programming-type problem irrespective of the method used are as follows:

1. *Collect problem information and data*

The first step is to collect information and data relative to the problem. Accurate facts and figures are needed plus estimates where accurate values are not available. The specific information required will vary with the objective and the problem to be solved. Under all circumstances a convenient unit of measure should be established for expressing problem data and information.

2. *Arrange problem information in an orderly manner*

The arrangement of information and data in an orderly manner facilitates understanding and solving the problem. Arranging information is a two-step procedure. The first step is to set up information in table form in order to clarify the relationships and conditions involved. This

step enables the problem to be "seen" and frequently reveals information that is as valuable as the solution itself. The second step is to set up the prepared information in an array of rows and columns for solution by one of the programming methods.

3. *Begin with a first solution*

Each linear-programming method specifies procedures which, if followed, provide a first solution to the problem as a place to start and the steps for improving it.

4. *Test the program*

Linear-programming methods provide procedures for testing each program to determine whether or not a given program or solution can be improved.

5. *Improve the program until the best or optimum has been calculated*

When the test indicates that a program can be improved, programming methods provide procedures and steps for revising each program which assure that each revision is at least equal to an improvement to the preceding program. Succeeding revisions are made until the test indicates no further improvement is possible—that an optimum or best solution has been reached. A "can do" program has been calculated from the "cannot do" restriction.

6. *Interpret the final program*

The final program provides specific answers and the steps to be taken to obtain the results that the best answer indicates are possible.

Because each of these basic steps involves a routine that is reducible to simple procedure and rules, it is possible to delegate the computation. In other words, the routine permits delegation of the computations and responsibility for results because rules and procedures are provided to guide action.

The Simplex Method is the classical and fundamental method. As a method it provides a way of using approximations in business computations. For example, the simplex method will compute the most desirable program and course of action involving such conditions as the following:

"We cannot buy more than 1,800 tons of sheet steel in the second quarter."

"We must make at least 800 of the economy model this month."

"The weaving department has available 2,900 loom weeks for production in the next period."

Generally, the simplex is the most reliable of the methods. It is, how-

ever, time-consuming and prone to error except when solved by an electronic computer.

The Transportation Method, sometimes referred to as the Distribution Method, is a specialized adaptation or modification of the simplex method. It was conceived and developed to solve transportation-type problems involving the distribution of a single product from several sources to meet requirements at a number of destinations at least transportation cost. The transportation method requires the expression of all problem information in comparable units and an equality between the amount demanded or required at various destinations and the amount that can be supplied from the various sources. As a specialized method, the transportation method makes it possible to work out the answers to a certain type of problem more simply and effectively than the simplex allows.

Experience gained in application has produced refinements and modifications to the transportation, or distribution, method and has widened its area of use. As a matter of fact, the transportation method and the modifications to it can be used to solve certain kinds of problems which are in no way related to transportation and distribution. The refinements and modifications have been grouped together and named the Modified Distribution Method, or Modi Method, which has become so highly developed that it is replacing the transportation method.

The Ratio Analysis Method, a recently developed method, shows that the basic process of solving a problem by linear programming is done by selecting and solving the heart or core of the over-all problem The ratio analysis method also provides a way of decreasing the number of computations necessary for solving some kinds of linear-programming problems. As such it is helpful in understanding the simplex method and is useful in and of itself as a problem-solving method.

There are times when the problem is too large or too time-consuming to solve by one of the precise methods. Then the objective is to find a way of approximating the best solution. The Index Method has proved particularly effective in this type of situation. The program developed may not be the mathematically best program, but experience has shown that it will be close to it. Because certain problems can be worked out in a fraction of the time and/or cost that would be required otherwise, the index method has proved extremely useful in practice.

As a matter of presentation, we believe it desirable to present the concepts and computational steps separately from the mathematical theory. We believe it is possible to understand and use linear programming without going through the mathematical derivation and underlying theories, in the same sense that a slide rule can be used without knowledge of logarithms. For this reason and for ease of understanding by

people in business who have long since forgotten their mathematics, we have elected to separate the concepts and procedural steps from the theory. This in no way prevents readers from obtaining both in this text. The theory is contained in the appendix. Section Two of the book therefore presents the underlying principles and computational steps of the various methods from the point of view of their application.

CHAPTER 2

The Transportation or Distribution Method

In this section we shall begin the presentation of the various linear-programming methods. The computational method to be discussed in this section is called the Transportation or Distribution Method. This section will include the following:

1. The background and origin of the transportation method
2. A transportation or distribution-type problem
3. The six basic linear-programming steps applied to the transportation problem
4. The procedures and rules for using the distribution or transportation method
5. The solution to the problem, including an alternate best program

BACKGROUND AND ORIGIN OF THE TRANSPORTATION METHOD

The transportation method—which is a special form of the simplex method—was originally developed to solve and provide planning information about transportation-type problems. The general problem was to determine the lowest transportation cost or distribution cost and program for shipping material between origins and destinations.

One of the first applications of this method was made in the oil industry. The problem was to distribute the output from a number of refineries to a larger number of supply points at least transportation cost. By using the distribution method it was possible to allocate a given amount of output to distribution centers at minimum cost and at considerable savings over former programs.

Although one of the first applications was to the distribution of refinery products, the general transportation-type problem solved resembles problems found in a variety of industrial situations, so that the method of solution has wide application to a number of industries and a number of problems that are not transportation problems. Both the transportation method and the modi method, which will be discussed in the next chapter, can be used to assign a given input, such as customers' orders, to a

given amount of facilities, such as machines or furnaces, in a way that will maximize profit margins or minimize costs.

A TRANSPORTATION OR DISTRIBUTION PROBLEM: THE COALCO CORPORATION

The quickest and simplest way to learn the transportation method is to solve a problem applying its special procedure within the framework of the six general steps for solving linear-programming problems as outlined in the introductory comments to this method.

1. Collect problem information

The problem to be solved and the required problem information is as follows:

THE COALCO CORPORATION

Least Transportation or Distribution Costs

PROBLEM: To distribute coal from mine (origin) to customer (destination) at the least transportation cost that will meet delivery promises.

GIVEN: 1. Three mines, located at Pittsburgh, Charleston, and Weirton, which provide 60, 35, and 40 tons, respectively, per week.

2. Five customers, A, B, C, D, and E, who require 22, 45, 20, 18, and 30 tons, respectively, during the week.

3. A schedule of transportation costs in dollars per ton by truck from each mine to each customer is shown in Table 2-1.

Table 2-1. Cost Table—Trucking Costs per Ton (In dollars)

Destination / Mine	Customer				
	A	B	C	D	E
Pittsburgh	−4	−1	−3	−4	−4
Charleston	−2	−3	−2	−2	−3
Weirton	−3	−5	−2	−4	−4

The information that has to be collected in order to solve the problem for least transportation cost is customer demand, available supplies for meeting demand, and the transportation costs per ton. They are considered fixed for the time period.

2. Arrange problem information in special table form

The second step in solving this problem by the transportation method is to arrange or array the amounts available and requirements data as shown in Table 2-2. This arrangement of supply-and-demand information is required for solution by the transportation method. Destinations and mines are arranged in the same order in which they are listed in Table 2-1.

Table 2-2. Requirements Table

Mine \ Customer	A	B	C	D	E	Tons available
Pittsburgh						60
Charleston						35
Weirton						40
Tons required	22	45	20	18	30	135 \ 135

Table 2-2 lists the amounts required at each destination in order to meet delivery promises to customers for the time period. As you can see, the Tons Required equals the Tons Available. This condition must always be present for the transportation method to apply. When it does not, procedures are available for getting around this requirement. They will be discussed, however, in the section dealing with the modi method. For the time being, we shall assume that this is a necessary condition that has to be met—at least in principle.

The intersection of each row and column represents a possible routing. An entry number at the intersection of Column A and the Pittsburgh Row represents the amount that would be shipped from Pittsburgh to Customer A. Similarly, the intersection of other rows and columns have the same meaning.

Table 2-1 shows the costs per ton in dollars incurred when each of the destinations or customers are supplied from each of the mines. One important point to be observed here is that there are alternative mines from which each customer can be supplied. The cost per ton in each case is shown at the intersection of the row and column involved. The unit costs in the table are preceded by a minus sign to indicate "costs." In other words, minus or negative values are to be thought of as costs.

Positive values would be thought of as profits if this were a greatest-profit solution. This interpretation is practical if it is assumed that costs "take away" while profits "add to" any position.

3. Develop a starting or first solution: the Northwest Corner Rule

Now that the data have been set up in table form, the next step is to find *some* solution to the problem as a place to start and then work to a better one.

Under the transportation method a systematic and logical procedure called the Northwest Corner Rule has been established for setting up the first solution. It is not absolutely essential to use this rule except that it does have the advantage of being systematic rather than rule of thumb or trial and error and does enable the method to be used much more effectively and permits the computations to be delegated to clerical personnel.

The general steps of the rule are as follows:

1. Start in the northwest or upper-left-hand corner of Table 2-2 and compare the amount available in a row with the amount required by the column.

2. Place the smaller of these two values in the column. If the value fills the customer requirement, then move to the right out of the northwest corner to the next column in the same row. The procedure is to exhaust the amount available at one supply point before electing to assign the amount available at the next supply point.

3. Compare the customer requirements with the amount available again and select the smaller of the two amounts. If the smaller amount represents the remainder of the amount available in a row, then that value is entered in the column and the remainder of the column requirement obtained by dropping down to the next row.

4. Repeat the same procedure outlined in steps 2 and 3, exhausting the amount available in each row before moving vertically to the next one and satisfying the column requirements before moving horizontally to the right to the next column.

With respect to the problem being solved, the NWC Rule involves assigning customers' orders to mines by starting in the upper-left-hand corner or northwest corner of Table 2-2 and working outward to the right and downward, in stair-step fashion, exhausting the amount available at each mine before supplying the next customer. Under these conditions then, the entries in each customer column must add up to the customer requirement shown in the bottom, or Requirements Row, in Table 2-2 if the customer is to be satisfied. Likewise, the entries in each row must add up to Tons Available at each mine since we cannot ship more than the mine can produce.

The NWC Rule provides us with a procedure for obtaining a first program and is used in computing better programs as well.

***Table* 2-3. First-program Table**

Mine \ Destination	A	B	C	D	E	Tons available
Pittsburgh	22	38				~~60~~ ~~38~~ 0
Charleston		7	20	8		~~35~~ ~~28~~ ~~8~~ 0
Weirton				10	30	~~40~~ ~~30~~ 0
Tons required	~~22~~ 0	~~45~~ ~~7~~ 0	~~20~~ 0	~~18~~ ~~10~~ 0	~~30~~ 0	135 / 135

In applying the NWC Rule to develop Table 2-3, we must perform the following steps:

1. Start in the upper-left-hand corner and compare the amount available at Pittsburgh with the amount required by Customer A. There are 60 tons available at Pittsburgh, while Customer A requires 22 tons.

2. Select the smaller of these two amounts—in this case 22 tons—and place it in the square formed by the intersection of the Pittsburgh Row and the Customer A Column. Circle the 22 and deduct it from the amount available at Pittsburgh, leaving 38 tons. Customer assignments are circled to prevent them from being confused with other numbers to be added to the table later.

3. Move to the right to the next column and determine the amount that Customer B requires. In this case it is 45 tons. Compare it with the amount available at Pittsburgh, which is 38 tons.

4. Select the smaller of the two amounts and place it at the intersection of the Pittsburgh Row and the Customer B Column. In this case it is the amount available at Pittsburgh, which leaves a deficit of 7 tons due Customer B that has to be obtained from another mine.

5. Obtain the 7 tons required to satisfy Customer B from the Charleston mine and deduct it from the amount available at the mine, leaving 28 tons available. Circle the value 7 and adjust the amount to zero at the bottom of the Customer B Column.

6. Move to the Customer C Column and again compare the amount required by the customer—20 tons—with the amount available at the mine—28 tons.

7. Select the smaller of the two amounts—in this case 20—and insert it in the square corresponding to the intersection of the Customer C Column and the Mine Row. Adjust values before proceeding.

8. Proceed in the same manner until all customer requirements have been satisfied and the amounts available at each mine are exhausted. When completed, the First Program should contain a series of stepping-stones or assignments starting in the upper-left-hand corner and proceeding in stair-step fashion downward and out to the lower-right-hand corner until all requirements have been met and all supplies used up.

It is advisable after the table is drawn up to make sure that the values total up horizontally and vertically. This gives a check on meeting customer requirements and using amounts available at mines.

After the check, the next step is to cost this program, since it is a program that will work—that is, customer requirements have been met and capacities of the mines have not been exceeded. The transportation or shipping cost of the First Program can be determined by referring to the original Cost Table (Table 2-1) and selecting the proper rate where orders have been assigned. If the applicable costs were checked they would form the same stair-step pattern as the circled values shown in the Requirements Table.

The cost of the program is determined by multiplying the tons assigned, indicated by a circled value in the First-program Table (Table 2-3), by the associated cost in Table 2-1. The sum of the multiplication gives the program cost. This is shown in the following calculations:

Cost of Shipments
(In dollars)

Pittsburgh Mine		Charleston Mine	Weirton Mine
22 × −4 =	−88	7 × −3 = −21	10 × −4 = −40
38 × −1 =	−38	20 × −2 = −40	30 × −4 = −120
		8 × −2 = −16	
Total	−126	−77	−160 = −$363

The First Program meets the general linear-programming conditions of starting with *some* program. The circled values provide a first feasible solution and also provide a basis or foundation upon which subsequent programs can be computed.

It is not always this simple, however, because there is a requirement of the transportation method which states that there must be a certain number of circled values in each program in order for the procedure to work. The certain number of circled values is established by a matrix algebra formula. Reduced to nonmathematical terms, this formula simply states that where there are *m* number of rows and *n* number of columns

in a matrix or table the number of entries or circled values have to add up to at least $m + n - 1$ entries. In the problem that we are solving there are three rows and five columns. This means that $m = 3$ and $n = 5$. The minimum number of entries needed to make the procedure work can be determined by substitution in the formula. Since $m = 3$ and $n = 5$, then the minimum number of circled values required is $m + n - 1 = 3 + 5 - 1 = 7$. Seven circled values are required to make the procedure work. Inspection of the first program shows that the requirement has been met.

As the computations progress there may be times when the total number of circled values will be fewer than $m + n - 1$. It happens sometimes during the computations that several of the values being manipulated are the same and "drop out" of the computations, and as a result the number of circled values becomes less than $m + n - 1$. The procedure in this case is to leave in the program one of the circles with a zero value in it and treat it as if it were a positive number. Eventually, it will adjust itself or drop out as the computations proceed.

4. Test the first program for improvement: rules for testing programs

At this point we want to know whether the First Program is the best, or least-cost, solution. This is the first point at which the true linear-programming steps become different from usual pencil-and-paper methods. This can be determined by examining each unfilled square in the table to see whether it is more desirable to move an assignment into one of them. The purpose of the examination is to see if a better combination of sources and destinations (mines and customers) can be developed. A better combination, if there is one, is developed by calculating a value with which to evaluate each vacant square relative to the existing program. Calculated values will provide the basis for determining if further improvement is possible or not (if there is a better combination of circled values than the one set up or not).

The rules for testing programs by evaluating vacant squares for possible improvement are as follows:

1. Select a vacant square to be evaluated. The important point to remember in evaluating vacant squares is that each square in the table represents a possible combination of source and destination. Further, any transfer of assignments from one source to another must be balanced by an offsetting transfer at another source.

2. Trace a path through the existing program. The evaluation process requires tracing a path through the existing program of circled values. The general rule for tracing a path is to start horizontally from the vacant square and using circled values as stepping stones, work back to the same vacant square using a series of vertical and horizontal

moves. Always select circled values that permit changing path direction; otherwise it will not be possible to work back to the vacant square being evaluated. Always follow the most direct route possible. Jumping over circled values is permitted.

3. Trace the *same path* in Table 2-1, setting out the costs or rates that correspond to the circled values. Attach a plus sign to the first circled value and then alternate minus and plus signs, stopping at the circled value in the column containing the square being evaluated.

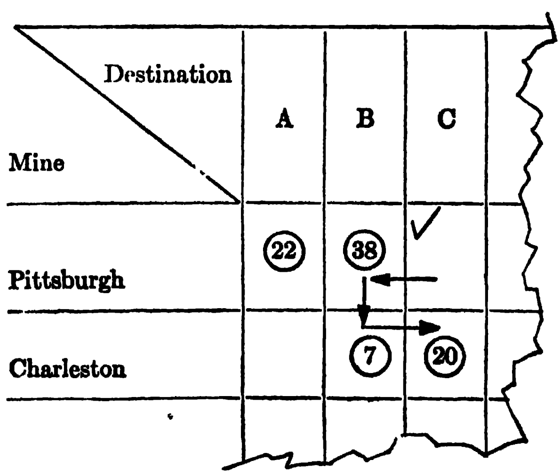
4. Total the plus and minus cost values thus obtained and then subtract from this total the cost value of the square being evaluated.

5. Repeat this process until all vacant squares contain a value. If all noncircled values are positive, the best answer has been obtained. If any of the values are negative, it means that a better solution is possible.

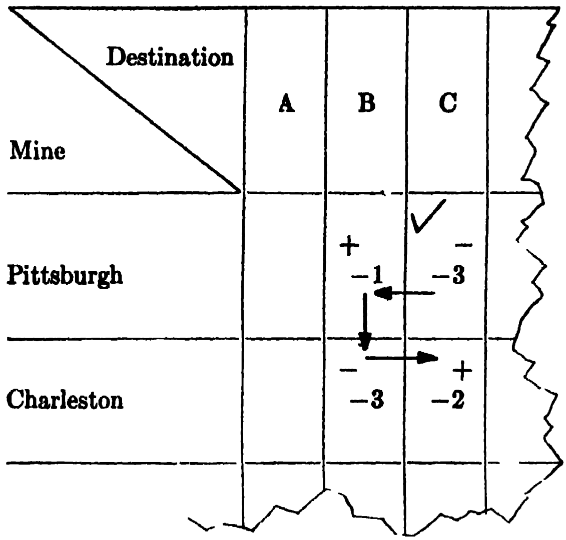
Now that we have the rules for testing a program, the next step is to test the First Program for transporting coal by computing values for the vacant squares of Table 2-3. Using the rules as a guide and taking them in order, we obtain the following results:

1. Select the empty square at the intersection of the Pittsburgh Row and the Customer C Column as a place to start. The square is designated in the table by a check mark. It makes no difference which vacant square is used as a starting point. Choose a circled value in the same (Pittsburgh) row which will allow a vertical, or 90-degree, change-of-direction move to another circled value. The circled value in the same row that will allow a vertical move is (38), which will allow a next move to (7).

2. Move to another circled value in the *same* row so as to return to the column containing the empty square being evaluated. The next value then in the same row as (7) is (20), and in this case it is in the same column as the square being evaluated. So the path traced out in Table 2-2 is as shown in the following diagram:



3. Trace the same path in Table 2-1. Reference to Table 2-1 shows that the path that corresponds will take in -1, -3, and -2.



To these values attach plus and minus signs, starting with a plus and alternating plus and minus signs as indicated.

4. Subtract from their total, which is $+(-1)-(-3)+(-2)=0$, the cost value of the square being evaluated. This result is $0-(-3)=3$, and the figure 3 is placed in the square at the Pittsburgh-Customer C intersection.

5. Repeat the process until all vacant squares are filled. When this has been done the First Program is as shown in Table 2-4.

Essentially, the evaluation process is carried out to determine the desirability of moving one of the circled values to a new position. Whenever any of the noncircled values are negative, a better answer can be obtained by moving *one* of the circled values from its present position to the square occupied by a noncircled value. If more than one negative value occurs, the most negative value designates the square to which a circled value should be moved.

Once it is determined that negative values exist, the next problem is to determine which circled value should be moved in order to obtain the best answer possible at that point. This is determined by retracing the path in both tables that was followed in establishing the square in which the most negative value appeared, selecting those circled values which were assigned a plus sign, and selecting the smallest value and using it as the point to start a new program.

Table 2-4. First Program

Mine \ Destination	A	B	C	D	E	Tons available
Pittsburgh	22	38	✓ 3	4	4	60
Charleston	−4	7	20	8	1	35
Weirton	−5	0	−2	10	30	40
Tons required	22	45	20	18	30	135

Total transportation costs: $363

The presence of negative values in the unassigned or vacant squares indicates that improvement is possible.

5. Improve the program until the best program has been calculated: steps for improving programs

The detailed steps for improving the solution until the least-cost solution has been calculated are as follows:

1. Retrace the path in both tables that was followed in establishing the square with the most negative value.
2. Select those circled values which were assigned a plus sign.
3. Select the smallest of those values and form a new table, placing that value in the square corresponding to the most-negative-number square.
4. Enter all other circles in the new table at their previous position but without any values.
5. Proceed to recalculate the next table, using the NWC Rule to determine the values to be inserted in the circles in the new table. Repeat the testing and evaluating procedure. Successive tables are calculated until all noncircled values are positive.

The steps for determining the circled value to move to improve the First Program start with the most negative number. This number is −5. Retracing the path in the program or matrix used to obtain that value involves the following circled values:

Destination / Mine	A	B	C	D	E
Pittsburgh	22 +	38 −			
Charleston		7 +		8 −	
Weirton	* −5			10 +	
Tons required	22	45	20	18	30

* Indicates square containing most negative number.

The corresponding path in Table 2-1 with alternate plus and minus signs added is as follows:

$$+ 10 - 8 + 7 - 38 + 22$$
$$+(-4) - (-2) + (-3) - (-1) + (-4)$$

The smallest circled value preceded by a plus sign is 7. This value is placed in the Second Program Table (Table 2-5) in the square corresponding to the one occupied by the −5.

Next, all circles with the exception of 7 are placed in the Second Program Table (Table 2-5) in the *same positions* they occupied in the First Program Table (Table 2-3) *without* any values being shown in them. Then, the assignment process is begun all over again, requirements being matched with supplies and the entire process repeated until one program has all positive values in the empty squares.

The program of least transportation costs is obtained on the Sixth Program. The fact that all the values in the vacant or unassigned squares are positive indicates that the best program has been reached.

Each Program, including the First through the Fifth, is a feasible program that meets customer requirements and does not exceed available capacity. Each successive program shows a decrease in transportation costs from the $363 of the First Program to the $290 minimum cost of the Sixth Program. Successive Programs from the Second to the Sixth are given in Table 2-5.

6. Interpret the final program

In problems as small as this one, it is frequently possible to obtain the best answer by inspection. But in larger problems it is difficult if not impossible to see the answer, so that methods or procedures of this kind are extremely valuable not only for obtaining the answer but also for *knowing when it has been reached.*

Even in a problem of this size the split of the orders among the mines to satisfy customer requirements is not always obvious. Another interesting point about the final program is that some customers are provided coal from mines that are not the cheapest transportation source for that particular customer. Customers A and E, for example, are not supplied from the mine having the cheapest transportation cost to them. *Over all,* however, the least-cost program has been obtained in terms of the delivery promises because the additional cost incurred by supplying Customers A and E in this fashion has been more than offset by economies in supplying the other customers.

ALTERNATE BEST PROGRAMS

Alternate best programs are possible whenever the best program contains a zero in an unassigned or noncircled square. Profits will *not* be increased by using an alternate instead of the best program originally calculated. The alternate program, however, provides an additional plan for obtaining the same best result, thus enabling management to be more flexible because there are more best choices at its disposal.

In this case the Sixth, or Best, Program contains a zero value in the square at the intersection of the Charleston Row and Customer E Column. The zero means that nothing is gained or lost by changing the program at that point. In other words, the net change obtained by moving a value into the square will be zero. The "cost margin" is zero, but additional flexibility is obtained.

The procedure for obtaining an alternate best program is to treat the zero as if it were a negative number and go through the same steps that are followed for computing a better program. When this has been done, the Alternate Best Program—the Seventh Program—is as given in Table 2-5.

The cost of the Seventh Program turns out to be $290, which is the same as the cost of the Sixth Program.

SUMMARY

In summary, the transportation or distribution method involves the following procedure:

Table 2-5. Successive and Best Programs

Second Program

Mine \ Destination	A	B	C	D	E	Tons available
Pittsburgh	15	45	−2	−1	−1	60
Charleston	1	5	20	15	1	35
Weirton	7	5	−2*	3	30	40
Tons required	22	45	20	18	30	135

Transportation Costs
(In dollars)

Pittsburgh	Charleston	Weirton
15 × −4 = −60 45 × −1 = −45	20 × −2 = −40 15 × −2 = −30	7 × −3 = −21 3 × −4 = −12 30 × −4 = −120
−105	−70	−153

Total cost of program = −$328

Third Program

Mine \ Destination	A	B	C	D	E	Tons available
Pittsburgh	15	45	0	1	−1	60
Charleston	−1*	3	17	18	−1	35
Weirton	7	5	3	2	30	40
Tons required	22	45	20	18	30	135

Pittsburgh	Charleston	Weirton
15 × −4 = −60 45 × −1 = −45	17 × −2 = −34 18 × −2 = −36	7 × −3 = −21 3 × −2 = −6 30 × −4 = −120
−105	−70	−147

Total cost of program = −$322

Fourth Program

Mine \ Destination	A	B	C	D	E	Tons available
Pittsburgh	15	45	−1	0	−2*	60
Charleston	7	4	10	18	−1	35
Weirton	1	6	10	2	30	40
Tons required	22	45	20	18	30	135

Pittsburgh	Charleston	Weirton
15 × −4 = −60 45 × −1 = −45	7 × −2 = −14 10 × −2 = −20 18 × −2 = −36	10 × −2 = −20 30 × −4 = −120
−105	−70	−140

Total cost of program = −$315

Fifth Program

Pittsburgh	5	45	1	0	10	60
Charleston	17	4	2	18	1	35
Weirton	−1*	4	20	0	20	40
Tons required	22	45	20	18	30	135

5 × −4 = −20 45 × −1 = −45 10 × −4 = −40	17 × −2 = −34 18 × −2 = −36	20 × −2 = −40 20 × −4 = −80
−105	−70	−120
Total cost of program = −$295		

Sixth and Best Program

Pittsburgh	1	45	1	1	15	60
Charleston	17	3	1	18	0*	35
Weirton	5	4	20	1	15	40
Tons required	22	45	20	18	30	135

45 × −1 = −45 15 × −4 = −60	17 × −2 = −34 18 × −2 = −36	5 × −3 = −15 20 × −2 = −40 15 × −4 = −60
−105	−70	−115
Total cost of program = −$290		

Alternate Best Program

Pittsburgh	1	45	1	1	15	60
Charleston	2	3	1	18	15	35
Weirton	20	4	20	1	0	40
Tons required	22	45	20	18	30	135

45 × −1 = −45 15 × −4 = −60	2 × −2 = −4 18 × −2 = −36 15 × −3 = −45	20 × −3 = −60 20 × −2 = −40
−105	−85	−100
Total cost of program = −$290		

* Indicates square to be filled.

— Indicates cost.

1. Allocate a limited amount of resources (mined coal) among the factors (customers) competing for their use according to some best criteria (least cost)

2. Use a matrix or special table arrangement for setting up the problem to be solved.

3. Provide a solution to start with.

4. Proceed in an orderly way through the computation of successive programs until a best answer is obtained.

5. Calculate margins or weights which are used to develop successive programs and determine when the best program has been reached.

6. Present a specific program which if followed will result in the best answer being obtained

7. Allow alternatives to be explored.

Next we shall take up the modified distribution, or modi, method, which is a simpler and newer version or adaptation of the transportation or distribution method and consequently is replacing it in practice. The transportation method, however, serves as a good introduction to the programming process and to the modi method.

CHAPTER 3

The Modi Method

The Modi Method has proved to be one of the most practical methods in application. In this section we shall discuss the modi as a method and explain the programming procedure and rules to be followed.

This section will include the following:

1. The background and development of the modi method
2. The requirements for using the method
3. The general procedure and rules for using the modi method
4. The application of the modi method to other types of solutions
5. The advantages of the modi method to management and the limitations of the method

BACKGROUND AND DEVELOPMENT OF THE MODI METHOD

The Modi—or Modified Distribution—Method is the direct result of an effort to make linear programming methods more practical for use by people in industry. It is an addition and refinement to the Stepping Stone Method,[1] developed by A. Charnes and W. W. Cooper, and to the Transportation-problem Procedure,[2] developed by A. Henderson and R. Schlaifer.

The modi method was developed in an actual business situation. The management of a certain company needed a rapid, uncomplicated, and easily understood and applied method for providing information. The nature of the problem, the frequency with which it had to be solved, and the limitations imposed by equipment and personnel were such that the techniques then available were not completely satisfactory.

The problem was a planning problem caused by the fact that the sales department sold more than the mill could produce. Since future plans called for increases in mill capacity, management wished to continue soliciting orders and expanding their market. With the mill oversold

most of the time, it was necessary for the company to purchase certain items from other producers of the same product in order to satisfy their customers. The handling, storing, and shipping charges involved in this purchasing and reselling usually consumed whatever margin there was, so that it was not profitable business.

To provide a reasonable basis for planning, management adopted a general practice of filling customers' orders according to location. Orders from local customers were scheduled to the mill first. Then orders from customers in ever widening circles from the mill were scheduled until the capacity for the period had been assigned. Customers farthest from the mill were then supplied by purchases in distant markets.

While this method of planning seemed reasonable, it did not take into account many important factors which determined the profitability of the operation. The modi method was developed for the specific purpose of planning this production and purchase program in such a way that maximum profit could be realized.

Linear programming revealed that it was actually far more profitable to produce and ship from the home mill certain items demanded in distant markets—and absorb the freight involved—than to purchase these items. The rate of profit of the mill for making certain items more than offset the additional freight or handling involved in sending them to a distant market. The program calculated by the modi method showed that a substantial increase in profit could be obtained by using this new technique as a basis for planning.

In addition to providing management with most profitable programs of "make" and "buy" as a guide for planning, the modi method offered opportunities for examining the profitability of other courses of action. Programs were calculated for highest utilization of equipment and largest quantity of production for purposes of comparison. The results of these comparisons provided an interesting illustration of the importance of proper selection of objectives. Both the most-profit and the largest-quantity programs required maximum utilization of equipment. The profit from the largest-quantity program, however, was approximately 20 per cent less than the profit from the most-profit program.

BASIC CONCEPTS AND IDEAS UNDERLYING THE METHOD

We said earlier that there are two so-called "classical methods" of linear programming. They are the simplex and the transportation or distribution[8] methods. Actually, the transportation or distribution method was developed from the simplex as a special, simplified method. The

modi is a "transportation-type method" that has been made more versatile by refinements and additions developed through application.

There are a number of basic conditions that data must satisfy before the modi method can be used. The most important is the necessity for a common or *standard unit* of measure. All information to be used in a modi solution, such as profits, capacities, ratio of production, sales demand, and the like, must be expressed in terms of identical units, such as standard machine hours. In technical language this condition is termed *homogeneity*. Expressing all data in standard units establishes relationships among the various items of problem information so that direct comparisons can be made. This permits the use of the modi method in place of the more tedious manipulations of the simplex method.

A standard unit is a numerical measure or common denominator that relates problem information and data in terms of the solution desired. The standard unit of measure may have many forms and is frequently difficult to establish. For example, it may take the form of dollars per cubic foot of storage space per month, or it may be manufacturing margin per standard furnace hour, or the like. It usually takes time and effort to decide on the proper unit and express all information in its terms. Once it has been done, however, the savings in computational time more than offset the "make-ready" time taken to prepare the data for solution.

The key to establishing the standard unit for the modi is to find one resource, such as a machine group, a furnace, an assembly line, one period of time, or the like, to which all other groups, furnaces, lines, or time periods can be compared or measured. Then by use of the selected factor, all problem information is expressed in relation to it.

Another requirement of the modi method is that the *demand or requirements must equal the available resources*. Input must equal output, the machine hours required must equal the machine hours available, the tons required must equal the tons available, and the like. This condition is rarely found in a practical problem, but that fact will not prevent use of the method. There are simple devices, such as dummy products or machines, that are used to create the equality and satisfy the technical requirements of the method.

Like all linear-programming methods, the modi method is a technique for manipulating numbers. All data and problem information must therefore be expressed numerically. In some cases it may be necessary to use estimates or approximations—especially when one is dealing with intangibles. For example, a condition like the desirability of avoiding overtime must be expressed as a number which represents the penalty connected with overtime. This may not be definable exactly and may have to be estimated but must be expressed to be considered.

The modi method employs a device that has been used by management for many years—that of putting up the data and problem information in one table where they can be studied and evaluated. Quite frequently, this step alone leads to considerable improvement because relationships and alternate possibilities are more easily seen. In the case of the modi, *computations are simplified* and speeded up by placing all information in one place where it can be seen and worked with.

In summary, the basic general concepts and requirements of the modi method are as follows:

1. It is a transportation- or distribution-type method that has been refined and developed through actual application to the point that it is versatile and understandable.

2. The key to the use of the method is establishing a standard unit of measure. This is often done by using one factor as a standard to which all other factors of the problem are to be compared. This is a requirement for using the modi method.

3. Demands and requirements for use of resources must equal the resources that are available—even if fictitious demands or resources must be added in order to make them equal.

4. Problems that involve ranges of values and sequencing conditions tend to limit the versatility of the method unless other steps are taken to overcome or offset them first.

5. All data and problem information must be expressed in numbers so that they can be set up in a single tabular array or tableau for manipulation.

GENERAL PROCEDURE AND RULES FOR SOLVING PROBLEMS

An example problem will aid in understanding the procedures and rules of the modi method. In the pages to follow, a model problem is developed and solved by the modi method.

The place to begin any programming problem is to establish that it *is* a programming problem. This is done by referring to the list of basic requirements and conditions to be satisfied. This list is as follows:

1. Objective: What optimum solution is required? Most profit, least time, least cost, or the like. The objective must be clearly indicated.

2. Restrictions: What resources limit or restrict attainment of the objective?

3. Alternatives: Are there choices or alternative ways of using the limited amount of resources? Linear-programming methods select the best choice—but only where alternatives exist.

4. Interrelationships: Does a change in one variable affect the objective or some other variable?

5. Numerical expression: Can all facts and conditions be stated or expressed numerically in the form of linear or straight-line equations or inequations?

The modi method, like all programming methods, conforms to the steps of the programming procedure, which are as follows:

1. Collect problem information
2. Arrange information in an orderly manner
3. Begin with a first solution
4. Test the program
5. Improve the program until the optimum is reached
6. Interpret the final program

The problem selected to demonstrate the modi method has been kept small purposely so as not to obscure the presentation and development of the rules and procedures by details and computations. Actual problems of considerable size are included in Section Three, "Application," to provide opportunity to solve real problems of industry.

Keeping in mind the identifying characteristics and steps in the programming technique, assume that we have a small machine shop, called the ASM Company, with four similar but not identical machines.

The problem is to assign six orders to four machines to make the largest amount of profit in the period being considered. Because of tooling, one of the orders runs on only one machine The other five orders can be run on several machines. If each order is run on the machine best suited for it (the ideal machine), some machines will stand idle during the period and deliveries will not be made in time to meet promises to customers.

Information relating to the problem and necessary for its solution is given in Tables 3-1 to 3-7.

At this point, then, we are ready to begin.

Steps for identifying a programming problem

We must first establish that a programming problem exists. This is done by going through the list of requirements and conditions for use of programming methods.

OBJECTIVE

To assign Orders A, B, C, D, E, and F to Machines I, II, III, and IV so as to maximize profits.

This is a statement of the objective that management wants. Other objectives might be least cost or least time.

Table 3-1. Sales, Production, and Accounting Information

Sales demand		Production rates (in pieces per hour)				Accounting costs (in dollars per piece)				Selling price (in dollars per piece)
Order	In pieces	Machine I	Machine II	Machine III	Machine IV	Machine I	Machine II	Machine III	Machine IV	
A	200	18	20	10	16	2.50	2.00	4.00	3.00	5.10
B	150	45	50	25	40	1.20	1.00	1.70	1.50	2.00
C	400	9	10	5	8	.65	.60	.70	.80	1.50
D	600	27	30	15	—	4 00	3 00	4.50	—	6.80
E	420	9	10	5	8	3.00	2.60	3 20	3 40	4.00
F	150	—	75	—	—	—	1.80	—	—	3.00
		.90	1.00	.50	80					

↑
Standard machine

Table 3-2. Available Capacity
(In standard machine hours)

Machine	Maximum hours available	Per cent utili-zation	Effective available hours	Index	Standard machine hours
I	40	.95	38.0	.90	34.20
II	40	.97	38.8	1.00	38.80
III *	80	.80	64.0	.50	32.00
IV	40	.87	34.8	.80	27.84
Total					132.81

* Operates two shifts

RESTRICTIONS OR LIMITATIONS

There are two types of restrictions in this problem. One type is the capacity restrictions—there are only a limited number of machine hours available, and it is impossible to schedule all orders on the best machine, Machine II, and make delivery on the scheduled date. The other type of restriction is the demand—the number of pieces on each order is to be met exactly.

Table 3-3. Profit per Standard Machine Hour
(In dollars)

Order	Machine	Profit
A	I	2.60 × 20 = 52.00
	II	3.10 × 20 = 62.00
	III	1.10 × 20 = 22.00
	IV	2.10 × 20 = 42.00
B	I	.80 × 50 = 40.00
	II	1.00 × 50 = 50.00
	III	.30 × 50 = 15.00
	IV	.50 × 50 = 25.00
C	I	.85 × 10 = 8.50
	II	.90 × 10 = 9.00
	III	.80 × 10 = 8.00
	IV	.70 × 10 = 7.00
D	I	2.80 × 30 = 84.00
	II	3.80 × 30 = 114.00
	III	2.30 × 30 = 69.00
E	I	1.00 × 10 = 10.00
	II	1.40 × 10 = 14.00
	III	.80 × 10 = 8.00
	IV	.60 × 10 = 6.00
F	II	1.20 × 75 = 90.00

Table 3-4. Demand
(In standard machine hours)

Order	Standard machine hours
A	10.0
B	3.0
C	40.0
D	20.0
E	42.0
F	2.0
Total	117.0

Table 3-5. Hours per Order by Machine Group

Order	Pieces required	Hours per order			
		Machine I	Machine II	Machine III	Machine IV
A	200	$11\frac{1}{9}$	10	20	$12\frac{1}{2}$
B	150	$3\frac{1}{3}$	3	6	$3\frac{3}{4}$
C	400	$44\frac{4}{9}$	40	80	50
D	600	$22\frac{2}{9}$	20	40	—
E	420	$46\frac{2}{3}$	42	84	$52\frac{1}{2}$
F	150	—	2	—	-

ALTERNATIVES

There are alternate machines on which the orders can be produced. All but one order, Order F, has several alternative assignments, and most of the orders can be produced on all four machines at varying rates of production.

INTERDEPENDENCE OR INTERRELATIONSHIPS

With the exception of Order F, which can be preassigned to Machine II, the orders compete with each other for machine time. A transfer of any one order from one machine to another will have an effect on other orders that can be run on that machine, on the total production of the machines, and on the profit that results.

LINEARITY

The objective, relationships, and conditions of the problem can be expressed as a series of related equations containing first-degree powers only. The controlling elements in the problem—the machine capacity and the profit margin per unit—are linear. An increase in available hours brings a proportional increase in output and profit.

Looking at the problem, then, we see a group of orders (demands) competing for a limited amount of machine time (resources). To meet delivery promises it is necessary to run some orders on alternatives to the best machine on which they can be run.

The problem, then, is to select from all the possible assignments of orders to machines the combination that maximizes profit under the conditions of capacity and sales. As a basis for planning, we want to know the most profitable choice or program of all available choices or programs.

STEPS FOR SETTING UP AND SOLVING A MACHINE-LOADING PROBLEM: THE ASM COMPANY

Having established that the problem is a programming problem, one that can be solved by linear-programming methods, we are then ready to proceed through the steps for solving the problem.

1. Collect problem information

This step has been done. The tables of information that have been compiled indicate the nature and kind of information that have to be collected to solve this type of problem.

2. Arrange problem information in special table form

This is an important step in the solution of any problem. Even if we were not using a programming technique, it would be useful to make an orderly or tabular arrangement of data. The fact that we know there are a number of possible answers and that it is difficult to select the best answer—especially where the problems are large—indicates that an orderly arrangement of information will be helpful in the analysis of the conditions and limitations of the problem.

1. One step, putting up the information in tables, has been done. There are a number of additional steps that have to be taken, however, to convert the information to usable form for solution by the modi method.

The next step is to stablish a standard unit that expresses the relationship among all variables and conditions of the problem. There is no easy rule for establishing the standard unit. A trial-and-error process is necessary, and practice and experience are extremely helpful. In general, the unit will involve time, money, space, material, or the like and may be a compound unit, such as dollars per square foot per month.

Experience has indicated that a standard machine hour is usually a satisfactory unit for production planning. Other examples are demonstrated in later chapters. Once we have decided on standard machine hours as the unit of measure for this problem, the next step is to select one machine as a standard to which the remaining machines can be compared. In this problem, Machine II is selected as the standard machine and given an index number of 1.00. Any machine could have been selected, but Machine II is convenient because it is the most efficient and all orders can be run on it. Each machine is compared to Machine II and given a number that expresses the relative speed of the machine in terms of the standard machine.

Sometimes the relative speeds of machines are not the same for all

products. Then double comparisons, weighting, and the like are necessary to establish a usable relationship. If an incorrect index is used, it will show up when the final answer is converted to a machine load for each machine. It will then be necessary to adjust the indexes until they are correct and yield a proper solution. After a number of solutions have been worked out, experience will indicate the values that will be correct under the conditions of the problem. Changes in product mix generally require a new solution of the problem. The next step is to convert the information to standard machine hours. Available machine hours are multiplied by the index to convert them to standard hours. Demand is converted from number of pieces to standard machine hours by dividing by the production rate on the *standard* machine. Profit per standard machine hour for each order is determined for each possible machine assignment. It is calculated by subtracting the production cost per piece from the product selling price and multiplying by the rate of production for the *standard* machine.

In establishing costs of production, fixed costs incurred whether the machine is running or not need not be included. The accounting costs charged to each order consist of the direct labor, material, and expenses that can be allocated to the machine on which the order is run.

2. The second step in arranging the information, now that it is in usable form, is to set it up in a grid or table (see Table 3-6). To reduce the number of calculations and establish a rule that can be followed, machines are assigned rows in order of decreasing profit from top to bottom in the matrix, and orders are arranged in order of decreasing profit from left to right. This arrangement decreases the number of calculations required to solve the problem.

A formal mathematical requirement of the method is that capacity must equal demand, so a dummy product column is added to take up the slack or idle time. This product has zero profit and will not enter the solution. If there were insufficient capacity to meet the demand, a purchase row could be added to represent a vendor. Profit values for this row could be determined by subtracting the comparable cost of the purchased item from the selling price. Valuable information often comes from the use of dummy demand or vendors in the calculations. The value of additional sales or additional capacity can be determined in this way as a basis for management planning.

A small subsquare, or box, is placed in the square formed by the intersection of each order column and machine row. The profit per standard machine hour for running the order on the machine is placed in the appropriate subsquare.

3. Develop a first solution

At this point, the table or matrix is ready for loading, or assignment. Starting at the upper-left-hand, or northwest, corner, we assign the orders to the machines, completely using the capacity of one machine before passing on to the next. For example, Order D requires 20 hours, and there are 36.8 hours available on Machine II. Therefore, Square DII is loaded with 20 hours, leaving 16.8 hours of capacity to be assigned. Moving horizontally across the top of the table, we next assign Order A, which takes 10 hours of the remaining 16.8 hours. Square AII receives a circled entry of 10 hours. Of the remaining 6.8 hours on Machine II, 3 hours are assigned to Order B and 3.8 hours to Order E, which has to be completed on Machine I. The process is repeated until all orders are assigned and all capacity allocated. The remaining capacity of 15.84 hours is then assigned to Dummy Product. The result is a staircase of loaded squares that extends diagonally from the northwest to the southeast corner. The loaded values are circled to distinguish them from other numbers used in the calculations. The result of this process, shown in Table 3-6, is a feasible program from which a profit of $3,855.36 is possible. This profit can be quickly calculated by multiplying each circled value by the profit associated with its square and then summarizing.

We are now at the point where we want to test Program A-1 to determine whether it is the best and, if not, how we can do better. Again, there is a procedure for determining this.

4. Test the program: compute row and column values

The first step in testing the program is to compute row and column values and place them in their appropriate squares as indicated in the table. Row and column values are used to evaluate the desirability (in this problem, profitability) of moving an order or part of an order into a vacant square (running it on an alternative machine), thus improving the program and increasing profit.

There are two basic rules to remember in computing row and column values:

1. The sum of the row and column values for loaded squares must equal the value in the box of the square formed by their intersection.

2. A value for one row or column must be assumed as a place to start. A convenient place to start to compute row and column values is the first row value to the left and top of the matrix, assigning to it a value of zero. This is a simple rule that clerical personnel can follow.

Actually the selection of the starting rows and column makes no difference in the final answer. It also makes no difference what value is assumed as the starting value. The entire process is a comparative one.

We are only interested in knowing whether there is a difference when we compare figures, not in the figures that provide the difference. By starting with a zero, we are in effect setting the profit of the existing program at zero and allowing the row- and column-value calculations to tell us whether the profit can be increased.

Starting in the upper-left-hand corner, we assign the row for Machine II a value of zero. We must remember to use loaded squares only, and the Row and Column values must add up to the value shown in the small box or subsquare. The Column value for Order D is 114, obtained as follows:

Let R be the Row value. In this case the value has been arbitrarily set at 0.

Let C be the Column value.

Let S or $\boxed{S}$ be the Square value.

Referring to Table 3-6, we find that Order D on Machine II earns \$114 per machine hour. So S, or this Square value, is \$114.

$$\text{Then} \qquad R + C = \boxed{S}$$

$$\text{and} \qquad S = \$114$$

$$\text{Therefore } R + C = \$114$$

But R has been arbitrarily given a value of 0 (zero).

Substituting

$$0 + C = \$114 \text{ or}$$

$$C = \$114$$

In the same way, the Column value for Order A is 62, the Column value for Order B is 50, and the Column value for Order E is 14. Because Order E has a Column value of 14 and Square EI has a Profit value of 10, the Row value for Machine I is $R + 14 = 10$, $R = -4$. With Row value of -8 and a Profit value of 7 shown in Square CIV, the Column value for Order C is 15. Similarly, we find that Row value for Machine III is -7, and the Column value for the Dummy is 7.

By following this general procedure, all Row and Column values can be computed. If they cannot be computed it means that the squares have not been loaded according to the rules or there are not enough loaded squares. The mathematical term used to describe the latter kind of breakdown in the calculations is called *degeneracy*. In other words, the process of working to a solution has degenerated *to the point where it is impossible to go on with* the computations.

Table 3-6. **Program A-1**

(One operator for each machine)

Machine	Col. / Row	D	A	B	E	C	Dummy product	Standard machine hours available
		114	62	50	14	15	7	
II	0	114 — 20	62 — 10	50 — 3	14 — 3.8	9 — 15	0 — 7	36.80 *
I	−4	84 — 110	52 — 58	40 — 46	10 — 34.2	8.5 — 11	0 — 3	34.20
IV	−8	☒	42 — 54	25 — 42	6 — 4	7 — + 23.84	0 — ✓ † −1	27.84
III	−7	69 — 107	22 — 55	15 — 43	8 — 7	8 — − 16.16	0 — + 15.84	32.00
Standard machine hours required		20	10	3	42	40	15.84	130.84 / 130.84

Changes in Load

C	Dummy product
9	0
8.5	0
7 — 8	0 — 15.84
8 — 32	0
40	15.84

* 2 hours deducted for Order F.
† ✓ Indicates improvement is possible.

Profit Program A-1

20	×	114	=	$2,280.00
10	×	62	=	620.00
3	×	50	=	150.00
3.8	×	14	=	53.20
34.2	×	10	=	342.00
4	×	6	=	24.00
23.84	×	7	=	166.88
16.16	×	8	=	129.28
15.84	×	0	=	0.00
				$3,765.36
Order F				90.00
Total				$3,855.36

Again there is a rule to serve as a guide in this situation. It states that **for a matrix of *R* Rows and *C* Columns there must be $R + C - 1$ entries or circled values.**

In a modi solution, if there are not sufficient circled values, the step to overcome degeneracy is to add a circle—generally where one existed in the preceding solution—and place a zero in its center. With all Row and Column values calculated, we are then ready to determine whether a better assignment can be made by testing the value of using a vacant or unloaded square.

Again, this can be reduced to rules. The procedure is to compare the sum of the Row and Column values to the value in the subsquare or box of each vacant square. *The basic purpose is to determine whether improvement or increase in profit can be obtained by moving an entry into a vacant square.*

The first rule is as follows:

If the sum of the Row and Column values is larger than the Profit value, it signifies that a greater profit is obtained by leaving that square vacant. If, on the other hand, the Row and Column values total less than the Profit value of the square, it means that profits can be increased by transferring an order to the vacant square (machine in the row of the square).

The second rule is as follows:

Start at the upper-left-hand corner and evaluate the vacant squares working outward and downward. At this point there are two options. You can stop at the first square in which the calculations indicate an improvement is possible. Or you can evaluate all vacant squares.

In Program A-1, the sums of Row and Column values are shown in the vacant squares. The evaluation is made by comparing these figures with the Profit values. Comparison shows that shifting orders into the squares formed by Dummy IV (0 is larger than -1) or EIII (8 is larger than 7) will increase profits. *This means that it is possible to do better and increase profits by changing assignments of orders to machines. The squares involved indicate the point at which improvement can be made.*

Profit will be increased by transferring work to either of the squares. The amount by which the Profit value in the square exceeds the sum of the Row and Column values is equal to the amount of additional profit that will be made by transferring 1 hour of work to the square.

5. Improve the program until the best program has been calculated

The program is improved by transferring work to the selected square. For this problem we shall select the Dummy III Square because it provides an opportunity to show how the Dummy can be helpful. The first rule for improving the program is as follows:

Starting in Square Dummy III, move left or right to a circled value from which a vertical move can be made. Then move up or down to a circled value from which a horizontal move can be made. Continue this pattern, alternating horizontal and vertical moves until the path is retraced back to the starting square. In this problem the path is the rectangle indicated by the arrows. In some cases the path will be more complicated. In every case the first move should be a horizontal one, and the aim should be to trace a path back to the starting square in the shortest number of steps, jumping over circled values when encountered.

The second rule is as follows:

Label the starting box with a minus (–) sign and the others along the path alternately plus (+) and minus (–). From the squares having a plus (+) sign, select the one with the smallest circled value. In this problem the value is 15.84 shown in Square Dummy III. Add this value to each of the squares having a minus sign and subtract it from each square having a plus sign. This always unloads or empties one square. It may happen that the smallest circled value may appear in several squares that have been given a plus sign. In this case, subtracting the value from the plus squares will empty several squares. As a result it will not be possible to compute Row and Column values because there are no longer $R + C - 1$ entries or circled values. To overcome this, select one of the plus squares to be emptied and leave it empty. In the remaining emptied squares, place a circle with a zero in it. The circled zero will serve as an entry for computations but will not affect the balance because it is zero and does not take machine hours or help to fill an order.

The new loading is shown in Table 3-7. The profit is now $3,871.20 —an improvement of $15.84. This proves that the move has improved the profit position. The improvement can also be shown by taking the difference between the profit in the subsquare and the sum of the Row and Column values $[0 - (-1) = 1]$ and multiplying it by the number of hours shifted: 15.84 = $15.84.

At this point we do not know whether the best or most profitable program has been obtained. Therefore, new Row and Column values must be computed and used to test the vacant squares for possible improvement. If improvement is indicated, then the load is shifted by tracing a path, selecting a value to be transferred, and balancing hours available with hours required. If no improvement is indicated, then the best answer has been reached in the calculations.

Applying this general process to Program A-1, we find that further improvement is possible. The new Row and Column values indicate that profit margin can be increased by transferring part of Order E to Machine III. The steps and results of the transfer are shown in Table 3-7.

Table 3-7. Program A-2

Machine	Row \ Col.	D	A	B	E	C	Dummy product	Standard machine hours available
		114	62	50	14	15	8	
II	0	114 20	62 10	50 3	14 3.8	9 15	0 8	36.80
I	−4	84 110	52 58	40 46	10 34.2	8.5 11	0 4	34.20
IV	−8	☒	42 54	25 42	+ 6 4	− 7 8	0 15.84	27.84
III	−7	69 107	22 55	15 43	−✓ 8 7	+ 8 32	0 1	32.00
Standard machine hours required		20	10	3	42	40	15.84	

✓ Indicates improvement possible.

Changes in Load

E	C
14 3.8	9
10 34.2	8.5
6	7 12
8 4	8 28
42	40

Profit Program A-2

20	×	114	=	$2,280.00
10	×	62	=	620.00
3	×	50	=	150.00
3.8	×	14	=	53.20
34.2	×	10	=	342.00
4	×	6	=	24.00
8	×	7	=	56.00
15.84	×	0	=	0.00
32	×	8	=	256.00
				$3,781.20
Order F				90.00
Total				$3,871.20

An improvement of $15.84

Table 3-8. Program A-3: The Most-profit Program

Machine	Col. / Row	D	A	B	E	C	Dummy product	Standard machine hours available
		114	62	50	14	14	6	
II	0	114 (20)	62 (10)	50 (3)	14 (3 8)	9 14	0 6	* 36.80
I	−4	84 110	52 58	40 46	10 (34 2)	8 5 10	0 2	34.20
IV	−7	☒	42 55	25 43	6 7	7 (12)	0 (15 84)	27.84
III	−6	69 108	22 56	15 44	8 (4)	8 (28)	0 0	32.00
Standard machine hours required		20	10	3	42	40	15 84	130.84

* 2 hours deducted for Order F.

Profit Program A-2

20	×	114	=	$2,280.00
10	×	62	=	620 00
3	×	50	=	150.00
3.8	×	14	=	53.20
34.2	×	10	=	342.00
4	×	8	=	32 00
12	×	7	=	84.00
28	×	8	=	224.00
15.84	×	0	=	0.00
				$3,785.20
Order F				90.00
Total				$3,875.20

An improvement of $4.00

The same process is repeated to arrive at Table 3-8. Program A-3 is the most-profit program because none of the vacant squares has a Profit value higher than the sum of its Row and Column values. This indicates that the most-profit program has been calculated. Frequently there are alternate best solutions. These occur when the sum of the Row and Column values exactly *equals* the value shown in the subsquare of the vacant square. By treating the condition as if it were one that could be improved, an alternative program for obtaining the same result will be provided. The profit will not be increased, but management will have increased flexibility for planning because it has another program for obtaining the maximum profit.

6. Interpret the final program

Now that the best program has been computed, the last step is to convert from standard units to original units and set up a machine load. The conversion steps are shown in Table 3-9.

Table 3-9. Conversion and Comparison of Computed Program to Original Problem Requirements

Order	Machine	No standard hours	Index	No actual machine hours	Pieces per hour	Pieces scheduled	Pieces required
A	II	10	1 00	10 00	20	200	200
B	II	3	1 00	3 0	50	150	150
C	III IV	28 12	.50 .80	56 0 15 0	5 8	280 120	400
D	II	20	1 00	20 0	30	600	600
E	I II III	34 2 3 8 4 0	.90 1.00 .50	38 0 3.8 8.0	9 10 5	342 38 40	420
F	II	2	1 00	2 0	75	150	150

The sales requirements are met by this program. Orders C and E are split and run on several machines as indicated in the table.

The machine load in actual hours is shown in Table 3-10.

All machines except Machine IV are fully loaded. Machine IV has 19.8 hours available.

Table 3-10. **Machine Load**
(In hours)

Order \ Machine	I	II	III	IV
A		10		
B		3		
C			56	15
D		20		
E	38	3.8	8	
F		2		
Total	38	38.8	64	15
Hours available	38	38.8	64	34.8

At this point, then, we have progressed through the basic steps of the programming approach, demonstrated the conditions and requirements that have to be satisfied to use a linear-programming method to solve a problem, and worked through the steps in the modi method, including the interpretation of the results. The steps for solving a problem by the modi method of linear programming are summarized below.

SUMMARY OF STEPS FOR APPLYING THE MODI METHOD

1. Express the problem and objective accurately.
2. Collect and analyze problem information.
3. Express the problem information and data numerically in common units.
4. Rearrange the data in a matrix or table so that the Northwest Corner Rule can be used to make assignments.
5. Begin with a first solution by making the basic assignment of circled values to squares using the Northwest Corner Rule. Evaluate the program as set up.
6. Compute row and column values, starting with zero in the first row in the upper-left-hand corner.
7. Test the program for improvement by evaluating the open squares to determine whether a better assignment can be made.

8. Transfer an assignment or part of an assignment to a more beneficial or profitable vacant or open square found in step 6.

9. Compute new row and column values.

10. Continue to improve programs by using the same basic procedure until further improvement cannot be obtained. At this point the value of each vacant square is equal to or less than the sum of its row and column values.

11. Interpret the final program.

ALTERNATIVE SOLUTIONS BY THE MODI METHOD

The problem just considered illustrates the use of a dummy column to dispose of *excess* resources. Insufficient resources or excess demands can be handled in the same way by using a dummy row.

1. Insufficient resources or excess demand

Suppose the sales demand for Order F in the period had increased to 1,800 pieces. Order F runs on Machine II only (index 1.00). At a production rate of 75 pieces per hour it would now require 24 hours to complete the order. This additional production requirement raises the demand for standard machine hours to 139, which exceeds the 132.84 standard machine hours available. To accommodate this condition and meet the technical requirements of the method the Dummy Machine Row is added.

The problem as set up in tableau or matrix form appears in Table 3-11.

Determining the values in the subsquares presents a new problem. The subsquare values for Dummy Machine must represent the profit from *not* making the item, which will usually be some negative value. For example, if we decided that some items *must* be made, the profit from *not* making them would be called $-M$. ($-M$ is a prohibitive penalty in such cases.) M is an indefinitely large number. It is not assigned a value but otherwise acts as if it were a number. It is treated as if it were a number whenever it enters the calculations. Actually it is a convenient mathematical device which has considerable use in programming procedure, as we shall see later in application.

In general, the $-M$ is used to prevent a certain event from happening in the calculations. In other words, the penalty or cost attached to the event is so large that the computations exclude the possibility of ever including the event in the final solution and program. It may help to think that $-M$ means "must not occur."

In the present problem, the method uses the $-M$ to make sure that all existing capacity is used in the most profitable way possible. Since

Table 3-11. Program B-1

Machine	Col. / Row	D	A	B	E	C	Standard machine hours available
		114	82	70	40	41	
II	0	114 14 80	62 82	50 70	14 40	9 41	14.80 *
I	−30	84 5 20	52 10 00	40 3 00	10 16 00	8 5 11	34.20
IV	−34	$\boxtimes$	42 48	25 36	6 26 00	7 − 1 84	27.84
III	−33	69 81	22 49	15 37	8 ✓ 7	8 + 32 00	32 00
Dummy machine	$-M$ −41	$-M$ $-M+73$	$-M$ $-M+41$	$-M$ $-M+29$	$-M$ $-M-1$	$-M$ 6.16	6.16
Standard machine hours required		20	10	3	42	40	115 / 115

* 20 hours deducted for Order F.

✓ Indicates improvement is possible.

Change in Load

E	C
14	9
10 16.00	8.5
6	7 27.84
8 26.00	8 6.00
$-M$	$-M$ 6.16
42	40

Profit Program B-1

14.8	×	114	=	$1,687.20
5.2	×	84	=	436.80
10	×	52	=	520.00
3	×	40	=	120.00
16	×	10	=	160.00
26	×	6	=	156.00
1.84	×	7	=	12.88
32	×	8	=	256.00
Total				$3,348.88

Table 3-12. Program B-2

		D	A	B	E	C	Standard machine hours available
Machine	Col. / Row	114	82	70	40	40	
II	0	114 / 14.80	62 / 82	50 / 70	14 / 40	9 / 40	* 14.80
I	−30	84 / 5.20	52 / 10.00	40 / 3.00	10 / 16.00	8.5 / 10	34.20
IV	−33		42 / 49	25 / 37	6 / 7	7 / 27.84	27.84
III	−32	69 / 82	22 / 50	15 / 38	8 / 26.00	8 / 6.00	32.00
Dummy machine	$-M$ −40	$-M$ / $-M+74$	$-M$ / $-M+42$	$-M$ / $-M+30$	$-M$ / $-M$	$-M$ / 6.16	6.16
Standard machine hours required		20	10	3	42	40	115 / 115

Profit Program B-2

14.80 × 114 =	$1,687.20
5.20 × 84 =	436.80
10.00 × 52 =	520.00
3.00 × 40 =	120.00
16.00 × 10 =	160.00
27.84 × 7 =	194.88
26.00 × 8 =	208.00
6 × 8 =	48.00
Total	$3,374.88

An improvement of $26.00

the Dummy Machine does not exist except in the calculations, we want to be sure that all of the real capacity is used. The $-M$ forces the orders, taken as a group, to their most profitable machines. The fact that there is no place for the least profitable work to go is the reason the excess demand ends up on the Dummy Machine. The work scheduled on the Dummy Machine is held over until the next period or subcontracted. The most-profit program under problem conditions is shown in Table 3-12. The most profit is $3,374 88. Comparing this total with the most-profit program—Table 3-8 in the previous problem—we see that selling the additional amount of Order F has reduced profits and caused a partial rather than a complete delivery on Order C. This is summarized in Table 3-13. With this kind of information, management is in a position

Table 3-13 Effect of Changing Order F

Program	Most profit (in dollars)	Number pieces Order C produced
A-3	3,875 20	400
B-2	3,374 88	338
Difference	500 32	62

to evaluate the profitability of orders and take action on promises to customers.

2. Multiple machine assignment [4]

In the original problem presented in this section, suppose that Machines I and IV are run by the same operator and that the operator can run only one at a time. Furthermore, let us assume that *either* machine can run the equivalent of 35 standard hours but that the operator can run a total of 50 standard hours only. One of the machines, therefore, will be idle for at least 20 standard hours. Handling this type of situation is fairly simple for the modi method, provided it is expressed correctly in the problem. The conditions that have to be included in the problem are as follows:

1. Schedule Machine I and Machine IV 35 hours each.
2. Produce 20 hours of Dummy Product on either machine and no other.

Table 3-14. Program C-1

Machine	Row \ Col.	D	A	B	E	C	Dummy product	Balancing dummy	Standard machine hours available
		114	62	50	14	15	7	$-M+7$	
II	0	114 20	62 10	50 3	14 3 8	9 15	0 7	$-M$ $-M+7$	36 80 *
I	−4	84 110	52 58	40 46	10 35 0	8 5 11	0 3	0 $-M+3$	35 00
IV	−8	☒	42 54	25 42	6 3 2	+ 7 31 8	0 −1	− √ 0 $-M-1$	35 00
III	−7	69 107	22 55	15 43	8 7	− 8 8 2	0 3 8	+ $-M$ 20	32 00
Standard machine hours required		20	10	3	42	40	3 8	20	138 80

*2 hours deducted for Order F.
√ Indicates improvement is possible.

Change in Load

C	Dummy product	Balancing dummy
9	0	$-M$
8 5	0	0
7 11 8	0	0 20
8 28 2	0 3 8	$-M$
40	3 8	20

Profit Program C-1

20 × 114 =		$2,280.00
10 × 62 =		620 00
3 × 50 =		150.00
3 8 × 14 =		53.20
35 × 10 =		350 00
3 2 × 6 =		19.20
31 8 × 7 =		222.60
8 2 × 8 =		65 60
3 8 × 0 =		0
		$3,760.60
Order F		90 00
Total		$3,850.60

Table 3-15. **Program C-2**

Machine	Row \ Col.	D	A	B	E	C	Dummy product	Balancing dummy	Standard machine hours available
		114	62	50	14	15	7	8	
II	0	20 $\boxed{114}$	10 $\boxed{62}$	3 $\boxed{50}$	3.8 $\boxed{14}$	15 $\boxed{9}$	7 $\boxed{0}$	8 $\boxed{-M}$	36.80
I	−4	110 $\boxed{84}$	58 $\boxed{52}$	46 $\boxed{40}$	35 $\boxed{10}$	11 $\boxed{8.5}$	3 $\boxed{0}$	4 $\boxed{0}$	35.00
IV	−8	$\boxtimes$	54 $\boxed{42}$	42 $\boxed{25}$	+ 3.2 $\boxed{6}$	− 11.8 $\boxed{7}$	−1 $\boxed{0}$	20 $\boxed{0}$	35.00
III	−7	107 $\boxed{69}$	55 $\boxed{22}$	43 $\boxed{15}$	−✓ 7 $\boxed{8}$	+ 28.2 $\boxed{8}$	3.8 $\boxed{0}$	1 $\boxed{-M}$	32.00
Standard machine hours required		20	10	3	42	40	3.8	20	138.80

✓ indicates improvement is possible

Change in Load

E	C
3.8 $\boxed{14}$	$\boxed{9}$
35 $\boxed{10}$	$\boxed{8.5}$
$\boxed{6}$	15 $\boxed{7}$
3.2 $\boxed{8}$	25 $\boxed{8}$
42	40

Table 3-16. Program C-3

Machine	Row \ Col.	D	A	B	E	C	Dummy product	Balancing dummy	Standard machine hours available
		114	62	50	14	14	6	7	
II	0	114 / 20	62 / 10	50 / 3	14 / 3.8	9 / 14	0 / 6	−M / 7	36.80
I	−4	84 / 110	52 / 58	40 / 46	10 / 35	8.5 / 10	0 / 2	0 / 3	35.00
IV	−7	☒	42 / 55	25 / 43	6 / 7	7 / 15	0 / ✓ −1	0 / 20	35.00
III	−6	69 / 108	22 / 56	15 / 44	8 / 3.2	8 / − 25	0 / + 3.8	−M / 1	32.00
Standard machine hours required		20	10	3	42	10	3.8	20	138.80

✓ indicates improvement is possible.

Change in Load

C	Dummy product
9	0
8.5	0
7 / 11.2	0 / 3.8
8 / 28.8	0
40	3.8

Table 3-17. Program C-4

Machine	Row \ Col.	D	A	B	E	C	Dummy product	Balancing dummy	Standard machine hours available
		114	62	50	14	14	7	7	
II	0	$\boxed{114}$ 20	$\boxed{62}$ 10	$\boxed{50}$ 3	+ $\boxed{14}$ 3.8	$\boxed{9}$ 14	$\boxed{0}$ 7	$\boxed{-M}$ 7	36.80 *
I	−4	$\boxed{84}$ 110	$\boxed{52}$ 58	$\boxed{40}$ 46	$\boxed{10}$ 35	$\boxed{8.5}$ 10	$\boxed{0}$ 3	$\boxed{0}$ 3	35.00
IV	−7	$\boxtimes$	$\boxed{42}$ 55	$\boxed{25}$ 43	$\boxed{6}$ 7	$\boxed{7}$ 11.2	$\boxed{0}$ 3.8	$\boxed{0}$ 20	35.00
III	−6	$\boxed{69}$ 108	$\boxed{22}$ 56	$\boxed{15}$ 41	− $\boxed{8}$ 3.2	+ $\boxed{8}$ 28.8	$\boxed{0}$ 1	$\boxed{-M}$ 1	32.00
Standard machine hours required		20	10	3	42	40	3 8	20	138 80

* 2 hours deducted for Order F.

Profit Program C-4

20	×	114	=	$2,280.00
10	×	62	=	620.00
3	×	50	=	150.00
3.8	×	14	=	53 20
35	×	10	=	350.00
3.2	×	8	=	25.60
11.2	×	7	=	78.40
28.8	×	8	=	230.40
3.8	×	0	=	0
20	×	0	=	0
				$3,787.60
Order F				90.00
Total				$3,877.60

An improvement of $27.00

To make sure that the method considers these conditions in the problem, there are several steps that must be taken:

1. Establish a second dummy column labeled Balancing Dummy to use up the 20 hours that the operator does not run.

2. Use $-M$s as profit values for the Balancing Dummy on Machines II and III. The $-M$ means that the idle time of the operator of Machines I and IV must not appear in the calculations on Machines II and III.

The initial program as it would be set up for computation is shown in Table 3-14. The Best Program, which is the Fourth Program calculated, is also given in Table 3-17. The programs in between are shown in Tables 3-15 and 3-16. The best program shows the following:

Machine IV is idle 20 hours.

Operator runs Machine I for 35 hours and Machine IV for 15 hours, a total of 50 hours.

The program meets the additional restrictions imposed by one operator running two machines.

The profit from this program is \$3,877.60.

3. Least cost

In each of the problems solved using the modi method, most profit was the objective for which a program or schedule was sought. As we have indicated a number of times, programming methods can be used to solve for a number of different objectives, and then the different results and programs can be compared as a basis for management planning.

If we were to solve the original problem in this section for minimum or least cost we would have to convert to cost per standard machine hour. The information about the selling price is not necessary for loading machine centers. We shall assume that all the work can be scheduled since capacity exceeds demand and that, therefore, the income from sales is fixed.

The arrangement of data follows the Northwest Corner Rule with the least costly orders, and the machines on which they run appearing in the upper-left-hand corner. Cost values go into the subsquares instead of the Profit values. The assignment of orders follows the staircase pattern.

There are two ways of treating the cost per standard machine hour:

1. One way is to leave all cost information positive, using the rule that *improvement is obtained by looking for vacant squares that have a lower cost value than the sum of the row and column values.* This time, instead of the zero profits that were assigned to the Dummy Machine, an M—a very large cost—is used instead. We cannot use zeros because that indicates no cost or the cheapest cost which would load the Dummy

Machine first. Instead we use *M* to make sure that there is no unscheduled work as long as there is capacity available. The arrangement of problem information in this way is shown in Table 3-18.

Table 3-18. Program D-1

Machine	Col. / Row	C	E	A	B	D	Dummy product	Standard machine hours available
		6.0	29.5	55.3	70 3	130.3	*M* − 4.7	
II	0	6 0 36 8	26 0 29 5	40 0 55 3	50 0 70 3	90 0 130 3	*M* *M*−4.7	* 36 80
I	5	6 5 3 2	30 0 31 0	50 0 55 8	66 0 70 8	120 0 130 8	*M* *M*−4 2	34.20
IV	4 7	8 0 10 7	34 2 11 0	60 0 10 0	75 0 3 0		*M* 3 84	27 84
III	4 7	7 0 10 7	32 0 34 2	80 0 60 0	85 0 75 0	135 0 20 0	*M* 12 0	32 00
Standard machine hours required		40	42	10	3	20	15 84	130 84

* 2 hours deducted for Order F.

The least-cost program is shown in Table 3-19.

2. The second way to look at the cost information involves the mathematical concept that minimization is a maximization process in which the key information has been multiplied by −1. Under this arrangement, the problem information is shown in Table 3-20.

The least-cost program is shown in Table 3-21. As you can see, there is no difference between the two least-cost answers.

ADVANTAGES AND LIMITATIONS OF THE MODI METHOD TO MANAGEMENT

The modi method is a practical, easy-to-work procedure that can be reduced to rules and taught to clerical personnel once the basic problem has been set up and routines established.

The modi method is self-correcting in the event that arithmetical mis-

Table 3-19. Program D-2

Machine	Col / Row	C	E	A	B	D	Dummy product	Standard machine hours available
		1	26	40	50	90	$M-7$	
II	0	6.0 / 1	26 0 / 3.8	40 0 / 10	50 0 / 3	90 0 / 20	M / $M-7$	* 36.80
I	4	6.5 / 5	30 0 / 34.2	50 0 / 44	60 0 / 54	120 0 / 94	M / M−3	34.20
IV	7	8.0 / 12	34 2 / 33	60 0 / 47	75 0 / 57		M / 15.84	27.84
III	6	7.0 / 28	32 0 / 4	80 0 / 46	85 0 / 56	135 0 / 96	M / M−1	32.00
Standard machine hours required		40	42	10	3	20	• 15 84	130 84

takes are made during the calculations. By balancing or adding up the assigned demands and assigned resources and comparing them to required demands and available resources, arithmetical errors can be corrected.

Perhaps the best advantages of the modi method are the savings in computation time and a reduction in the number of calculations. Both of these advantages stem from more efficient organization of problem information and greater simplification in the methods of calculating programs. These advantages represent real savings in time and money when problems and programs are required to be solved and calculated frequently. Also large problems can be worked out manually in a comparatively short time. These advantages make it possible to solve for several best answers—profit, cost, time—in a relatively short time so that management can consider a number of possibilities before making a commitment. In addition, alternatives can be explored and their effect assessed before the fact. In short, the manager has information that improves his decisions and effectiveness

These advantages tend to be offset in part by the fact that the modi is a specialized method and can be used only when problem data are set

Table 3-20. Program E-1

Machine	Row \ Col.	C	E	A	B	D	Dummy product	Standard machine hours available
		−6	−29.5	−55.3	−70.3	−130.3	−M +4.7	
II	0	−6.0 / 36.8	−26.0 / −29.5	−40.0 / −55.3	−50.0 / −70.3	−90.0 / −130.3	−M / −M+4.7	36.80*
I	−.5	−6.5 / 3.2	−30.0 / 31.0	−50.0 / −55.8	−60.0 / −70.8	−120.0 / −130.8	−M / −M+4.2	34.20
IV	−4.7	−8.0 / −10.7	−34.2 / 11.0	−60.0 / 10.0	−75.0 / 3.0	×	−M / 3.84	27.84
III	−4.7	−7.0 / −10.7	−32.0 / −34.2	−80.0 / −60.0	−85.0 / −75.0	−135.0 / 20	−M / 12	32.00
Standard machine hours required		40	42	10	3	20	15.84	130.84

* 2 hours deducted for Order F.

Table 3-21. Program E-2

Machine	Row \ Col.	C	E	A	B	D	Dummy product	Standard machine hours available
		−1	−26	−40	−50	−90	−M+7	
II	0	−6.0 / −1	−26.0 / 3.8	−40.0 / 10	−50.0 / 3	−90.0 / 20	−M / −M+7	36.80
I	−4	−6.5 / −5	−30.0 / 34.2	−50.0 / −44	−60.0 / −54	−120.0 / −94	−M / −M+3	34.20
IV	−7	−8.0 / 12	−34.2 / −33	−60.0 / −47	−75.0 / −57	×	−M / 15.84	27.84
III	−6	−7.0 / 28	−32.0 / 4	−80.0 / −46	−85.0 / −56	−135.0 / −96	−M / −M+1	32.00
Standard machine hours required		40	42	10	3	20	15.84	130.84

up in the same units. First applications indicate that it is not too difficult to do this, although it may take additional time. Once done, however, the reduction in computing time far offsets the extra time required to put the data into the same units.

Another limitation of the method is that it cannot accommodate several stages of a problem or situation simultaneously. For example, we cannot solve a production planning problem considering separately a sequence of operations, such as engine lathe, grinder, and assembly. They can only be handled separately or as a unit.

In some problems it may be desirable to work with ranges of numbers rather than with a definite specific number. The modi method cannot accommodate such situations.

Another condition which may limit the use of the modi, unless it is overcome by devices, is the inability to handle a number of problem restrictions and conditions at the same time. It is possible, for example, to handle certain sales and capacity restrictions at one time; but it is not possible to introduce sales, capacity, storage, space, material, and yield conditions into one matrix or problem setup at one time. There have not been many cases where this has been necessary, because with experience it is possible to work with the data so that once in the matrix it reflects the desired condition.

Possible areas of application for the modi method are many. The list is growing. The modi typifies and is leading and pointing the way to better, more practical applications of mathematics to business problems.

We shall return to the modi method in Section IV, "Application."

CHAPTER 4

The Simplex Method: Part 1

The Simplex Method is the basic linear-programming method from which the other known methods have been derived Basically, it is a technique for obtaining an optimum solution to a set of related linear equations and inequations. The method selects from all possible solutions the unique solution (or solutions) which makes a certain function (cost, quantity, profit, or other requirement) a maximum or minimum.

The simplex method is the universal method, being adaptable to the solution of a wider range of problems than any one of the other methods. Since it is fundamental to the field of linear programming, it will be treated in detail in this and the succeeding chapter.

The purpose of these chapters is to show *how to use* the simplex method, and they will therefore present the method as a practical technique for solving many industrial problems. The derivation and proof of the mathematics underlying the method are omitted from this section. They are, however, stated in Section IV, "Technical Appendix," under the title "Mathematical Discussion of the Simplex Method." Other references are contained in the Bibliography.

BACKGROUND AND INTRODUCTION TO THE SIMPLEX METHOD

In the previous section on the modi method, the method was introduced by describing the actual industrial problem that gave rise to its development. The steps and rules of the method were explained in terms of this actual problem. This type of explanation is convenient for illustrating key points. It also has the advantage of providing the reader with a specific reference when problems of application are encountered in practice.

The origin of the simplex method, however, was mathematical rather than industrial. As mentioned earlier, the development and background of the simplex has been primarily in the field of mathematical economics, and it is based on the work of Dantzig, Orden, Charnes, and their colleagues. There is, therefore, no "original industrial problem." But to

follow the practice of using industrial problems to explain and develop the essential steps of each method, a small but realistic hypothetical problem will be used in place of an actual one.

A SIMPLEX PROBLEM: THE BI-PRODUCT COMPANY

To solve a problem by the simplex method requires stating the desired objective as an equation in terms of the unknowns for which a solution is sought, expressing the conditions and restrictions as equalities and equivalent equalities, arranging the coefficients of the unknowns in a table or grid (matrix), and so constructing a first feasible solution that the first answer is zero. The first solution is then replaced by another feasible solution in which the answer is as large as or larger than the previous one. This replacement process continues in incremental, or step-by-step, fashion until the method indicates that further increases are not possible, thus indicating the attainment of an optimum solution.

Each replacement of a solution by a new solution is called an *iteration*. Each new solution is developed by selecting that activity (shown by a matrix column) which shows the largest unit margin or possibility for improvement and then bringing it into solution by replacing an activity which is less desirable. The final solution, obtained after a series of iterations, represents the best possible program under the specified restrictions and conditions. With this introduction to the steps of the method, we now turn to the problem.

The problem to be used to demonstrate the simplex method is a product-mix-type problem. A product-mix problem is perhaps the best problem for demonstrating the simplex procedure because it can be expressed simply and completely with few conditions and restrictions. This makes it possible to stress certain features of the method without complicating the picture unduly.

The statement of the problem and the information needed to solve it follow.

BI-PRODUCT COMPANY

Most Profitable Product Mix

PROBLEM: To determine the most profitable mix of two products—Product A and Product B. The problem assumes that all products made can be sold.

GIVEN: Three machine centers, the hours available on each during a 1-week period, the time per piece expressed as hours per piece, and the profit margin per piece. The profit margin per piece is provided by the Accounting Department and for purposes of the problem is essentially the difference between selling price and variable costs. Profit margin per

piece is considered to be firm for the period being considered. Fixed costs are considered separately and are not reflected in the data.

Products A and B have to be processed through all three machines in order to manufacture a completed piece.

Only complete pieces will be accepted.

The detailed information and conditions of the problem are summarized in Table 4-1.

Table 4-1. Production and Profit Information

Machine center	Hours per piece		Hours available
	Product A	Product B	
I	2	4	Up to 48
II	4	2	Up to 60
III	3	0	Up to 36
Profit margin per piece (in dollars)	6	4	

This problem, because of its characteristics and small size, can be solved graphically and algebraically, as well as by the simplex method. It can be solved by inspection as a few moments' study will show. Larger problems, typified by many products, machine centers, operations, rates of production and the like, cannot be shown graphically and are difficult to solve algebraically. Trying to solve them effectively by inspection requires either a crystal ball or more than a fair share of good luck. This point can be demonstrated easily by enlarging the given problem so as to include several more products and machine centers and then attempting to find a "best," or optimum solution. Such a solution is nearly impossible by the usual methods employed in most of industry

The problem used to demonstrate the simplex method is made simple to facilitate understanding of basic concepts, principles, and computational steps. Because of its small size, it is possible to see what is going on as the calculations are carried forward to the final answer. It can also be compared to a geometric solution in order to provide a clearer picture of what takes place. Then, if there is a question about what is happening in larger problems when they are encountered in practice, reference to the geometrical solution of the problem will be helpful.

GEOMETRIC OR GRAPHIC REPRESENTATION AND SOLUTION

As long as linear-programming problems are small they can be plotted geometrically. But when they are larger—beyond three dimensions (involving three products, for example)—they cannot be shown geometrically because there is no way of showing multidimensional space. Although no one has seen multidimensional space, or "*n*-dimensional space," as it is referred to technically, it is assumed that for simplex purposes a problem can be formulated and an answer obtained similar to that obtained for three-dimensional problems.

In the March, issue of *Mechanical Engineering,* R. M. Morse had an interesting explanation of this, as follows:

> The linear programming problem can be visualized most simply in geometrical terms. The *n* variables define *n* dimensional space; a point of this space corresponds to a hyperplane in this space, restricting the allowed solutions to one side of the hyperplane. By the time we have finished specifying all the restrictions (negative production not allowed, maximum limits on storage capacity, limits on production, and so on) we find that we have surrounded the region of possible solution by hypersurfaces, so that the allowed region is the interior of a convex polyhedron in the hypersurface. If the function to be optimized is a linear function of the variables, then the requirement that this function have some constant value also corresponds to a hyperplane which may or may not cut through the polyhedron; if it does, then it corresponds to an operationally possible value of the function to be optimized. By changing the value of the constant, we can generate a family of hyperplanes parallel to each other, their distance from the origin being proportional to the value of the function to be optimized.
>
> Some of the hyperplanes in this family cut through the polyhedron containing the region of solution; some do not. There are two limiting hyperplanes, one corresponding to the largest value of the function for which the hyperplane just touches the polyhedron, and one corresponding to the smallest value which just touches. Consideration of the geometry shows that for most orientation of the family of planes, the limiting planes just touch a vertex of the bounding polyhedron and thus contain just one possible solution compatible with all the boundary conditions. The outermost limiting point is the optimum solution if the function is to be maximized. Once the geometry is clear in one's mind, it is easy to visualize the solution. But, at present, it is not easy actually to compute the optimum vertex

when there are several dozen variables and about a hundred boundary faces of the polyhedron.

The problem chosen to illustrate the simplex method is two-dimensional. It can be plotted on rectangular coordinates with each product shown on a separate axis.

Geometrically, the solution to the problem must fall within the shaded

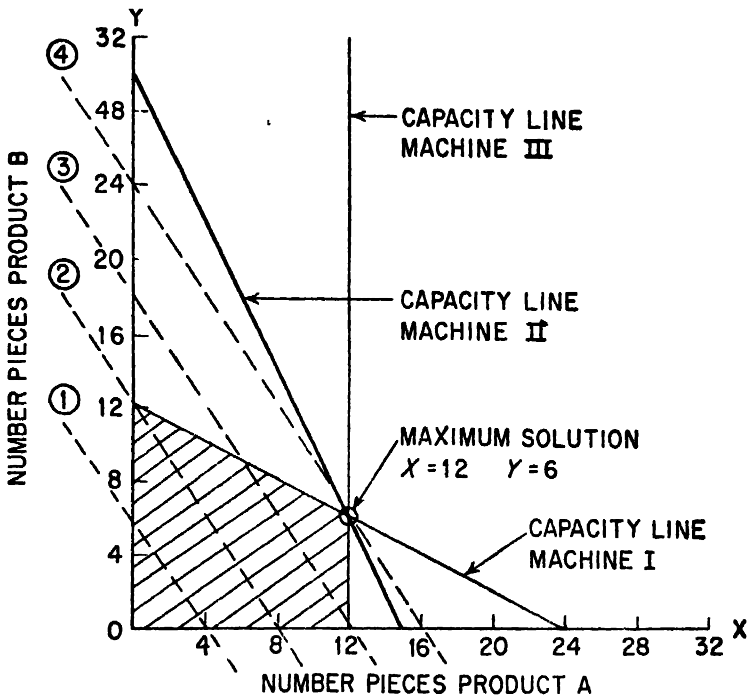


FIG. 4-1 Geometric Representation an Solution Most Profitable Product Mix for Bi-Product Company

area bounded by the three capacity lines. This means that the number of *completed* products cannot exceed the amount of capacity common to *all* machines. To be realistic the answer must be positive (real pieces) so that the solution must also lie in the first quadrant bounded by the X and Y axis, the only quadrant in which both X and Y are positive. Since *capacity* is expressed as an *equation* or line, the solution to the problem will occur at an intersection of the capacity lines shown in Figure 4-1.

The profit equation expressed as a line must be added to the geometric figure to obtain a profit answer. In this problem, the profit expression can be shown as a series of parallel lines moving outward from the origin. In larger problems, the profit expression becomes a plane which acts the same way to indicate most profit.

The most profitable answer to the problem is indicated geometrically at the point at which one of the profit lines touches the outermost part of the shaded area. At this point *all* of the shaded area lies between the profit line and the origin. Line ④ on the diagram is the line of most profit because all of the shaded area—the area common to all three machines—lies between the line and the origin. Line ③ on the other hand, although desirable and feasible, shows that there is capacity available which can be used for further production and profit.

ALGEBRAIC SOLUTION

The values and relationships expressed by the capacity equations of this problem have been selected with care in order to demonstrate that a solution to the problem is possible algebraically. The solution, however, is unique to the conditions of the problem. Very seldom does it happen that algebraic methods can be used to find an optimum point for this type of industrial problem—especially when the problem involves many unknowns.

An algebraic solution to the problem can be obtained by considering the equations of machine capacity, which are as follows:

$$\text{Capacity Machine Center I:}\quad 48 = 2X + 4Y \qquad (4\text{-}1)$$

$$\text{Capacity Machine Center II:}\quad 60 = 4X + 2Y \qquad (4\text{-}2)$$

$$\text{Capacity Machine Center III:}\quad 36 = 3X \qquad (4\text{-}3)$$

where X = amount of Product A
Y = amount of Product B

Equation (4-3) shows that $X = 12$, and by substitution in either Equation (4-1) or Equation (4-2) $Y = 6$. Therefore, the algebraic solution to the problem is $X = 12$, $Y = 6$, the same solution as obtained graphically. These values check out in Equations (4-1) and (4-2).

The algebraic solution was worked out to call attention to and demonstrate a number of features of the simplex method. First, to solve a problem by the rules for solving simultaneous equations requires at least as many equations as there are unknowns. For example, if there are 20 unknowns, it is necessary to have 20 equations to obtain the answer. In industry, many problems, such as machine-loading problems, involve hundreds of products and many machine centers. This would make the solution by simultaneous equations impractical because of the sheer size of the problem—even if it were possible to have as many equations as unknowns. The simplex method does not require as many equations as unknowns in order to calculate an answer which makes it more useful for industry.

Second, there is nothing implicit in an algebraic solution that provides an optimum or best answer. This can be illustrated by use of simultaneous equations. Suppose that the expressions of machine *capacity* are $6X + 2Y = 420$ and $5X + 5Y = 420$ and that the profit expression is $Z = 8X + 9Y$. The solution by simultaneous equations is $X = 63$ and $Y = 21$, which gives a profit of \$693. The most profitable answer by simplex or geometrical solution occurs at $X = 0$, $Y = 84$, which gives a profit of \$756.

STEPS AND REQUIREMENTS FOR SETTING UP A PROBLEM FOR SOLUTION

To solve a problem by the simplex method requires arranging problem information in a special way and following certain procedures and rules in calculating a solution. The general process for solving a problem by the simplex, once it has been identified, involves setting it up for solution, using the "mechanical" routines. These steps will be demonstrated using the product-mix problem of the Bi-Product Company.

To solve a problem using the simplex method demands that certain requirements are met and that certain conditions exist. These conditions are very important when it comes to setting up the problem, since correctly setting up the problem is essential to the success of the result. The mechanical techniques involved from there on are fairly simple and routine.

In order to set up the problem correctly the following conditions must be met:

1. There must be a known, definable objective, such as maximizing profit or output or minimizing time or costs (objectives such as these are also termed *criteria*). The mix that gives the highest profit margin is the objective sought in the problem under study.

2. There must be known, definable restrictions or limitations on the amount or extent of the attainment of the objective. Such limitations might be machine capacity, material restrictions, sales commitments, and storage limitations. The 48 hours available on Machine Center I is a typical capacity restriction that limits production and consequently profit for the period being considered.

3. The objective must be expressed in algebraic form in terms of unknowns (terms whose value is to be determined) with explicit coefficients. For example, highest profit margin $= 6X + 4Y$, where X and Y represent unknowns, the most profitable amounts of which are to be determined when the profit margin per unit is \$6 and \$4, respectively.

4. The restrictions or limitations on the attainment of the objective must be expressed algebraically and contain the unknowns expressed in the objective. These may be stated as equations or inequations. For example, the amount of Product A—denoted by X—and the amount of

Product B—denoted by Y—that can be produced are limited by the amount of time available on Machine II. Algebraically, this capacity restriction may be stated as $4X + 2Y \leqq 60$. This kind of expression is known as an *inequation*. This inequation stated in words says simply that the amounts of products to be run may or may not take all of the time available. This is an option that management wants to exercise in running the machine anyway and is therefore highly practical. In order to make the method and solution recognize the condition, it must be built into the problem as it is set up for solution. The inequation is the mathematical way to do it. To put it another way, if the expression read $4X + 2Y = 60$, it would mean to the method that all the 60 hours must be used to produce the two products, and the answer would come out accordingly. The answer, depending on which expression is used, may be quite different and serves to highlight the care that must be observed in formulating a problem for solution.

Another example of a restriction is illustrated by sales commitments for delivery of 20 pieces of Product B during the production period In other words, the required amount of Product B—denoted by Y—to be produced must equal 20 pieces ($Y = 20$). This restriction is an equation. Each of the expressions contains the variables or unknowns X and Y that are found in the objective.

In addition to these general conditions, there are a number of other requirements which must be met before the simplex method will apply. These basic requirements are as follows:

1. The various *activities* involved must be identified

The activities of a given problem refer to the various "actions" in which it is possible to engage. For example, an activity might be the manufacture of a product, the shipping of a product, the storage of material, and the like. When different products, such as Product A and Product B, are to be manufactured, each product represents a different activity. When the same product can be manufactured on alternate machines, *each* alternate method is an activity. The manufacture of a product on regular shifts is a different activity from the manufacture of the same product on overtime.

Generally, an activity is represented by a column when the problem information is set up in tabular or matrix form for solution.

2. The level of activity must be specific and measurable

For example, the Boston warehouse can store up to 500 packages, Machine I can be run up to 48 hours per week, Machine II can be run up to 60 hours, and the like.

3. The unknowns must be linear

This means that changing the *level* of the activity must change the *effect* of the activity proportionately. For example, the amount of profit and pieces produced in the sample problem must go up or down evenly with the level of production or number of available machine hours. This relationship when plotted in break-even-chart style will appear as a straight line. Stated another way, linearity means that a change in the time allotted to a product causes a proportional change in the amount of profit and product.

4. The various activities must be interdependent

Activities are interdependent when they must share limited amounts of resources which they use in common, when one activity produces a commodity which another uses, or when several activities each produce a commodity used by another activity.

The first type of interdependency exists when various products compete for available machine time so that the amount of each product produced depends on the time assigned to the others. The more of Product A that is manufactured, for example, the smaller the amount of Product B that can be made. This type of interdependency is the result of a *restriction* on machine capacity.

Another type of interdependency would occur if Product A required Product B in its manufacture. The amount of Product A that can be made is dependent upon the amount of Product B that is available.

5. The direct restrictions on the level of the activities must be stated in numerical terms

For example, a direct restriction is that at most 200 pieces of Product A can be sold this period, or the operating level of the Kansas City plant must be maintained at least at 60 per cent of rated capacity.

6. The objective or criterion must be a function of the level of each of the activities engaged in and must also be linear

For example, if the objective is to maximize profits, then the amount of profit depends upon the total *amount* of each product that is to be produced and the profit per piece of each. Further, an increase in the amount produced brings a corresponding or proportional increase in profit.

A summary of the conditions and requirements that must be met to use the simplex method is as follows:

1. The various *activities* must be identified.
2. The level of activity must be specific and measurable.

3. The unknowns must be linear.

4. The various activities must be interdependent.

5. The direct restrictions on the level of the activities must be stated in numerical terms.

6. The objective or criterion must be a function of the level of each of the activities engaged in and must be linear.

DESCRIPTION AND PARTS OF A MATRIX

A Matrix, or Tableau, is a form in which problem information is put up for computation. Before we go into the specific steps for setting up and solving a matrix, it will be helpful to describe a matrix and to identify the parts and function of each. Figure 4-3 shows the required orderly arrangement of numbers in rows and columns. These numbers are the constants and coefficients of the unknowns, collected and rearranged to facilitate computation.

The objective and conditions of the problem expressed as equations and inequations are as follows:

Objective: Most profit $Z = 6X + 4Y$

where Z = the maximum answer
X = no. of pieces Product A
Y = no. of pieces Product B
that provide the maximum Z.

Capacity restrictions on profit:

Machine I: $48 \geqq 2X + 4Y$

Machine II: $60 \geqq 4X + 2Y$

Machine III: $36 \geqq 3X$

These inequations state that the products that make up the most profitable mix can use all or part of ($\geqq$) the capacity of the machine centers.

When this information has been converted, collected, and rearranged in proper form, the result is the simplex matrix, or tableau, shown in Table 4-3.

There are two important points that must be taken into consideration before one sets up a simplex matrix.

1. All the equations and inequations *must* relate to the same problem.

2. Inequations (expressions of machine capacity in the example) must be converted to equivalent equations by the addition of idle, or "slack," variables which are also unknown variables. The satisfaction of these

requirements will be demonstrated in the following description of the matrix and its parts.

1. Parts of a matrix

The four principal parts of a matrix are the Column Caps, Stub, Body, and Base Row. These main parts together with their subparts are shown in Figure 4-2.

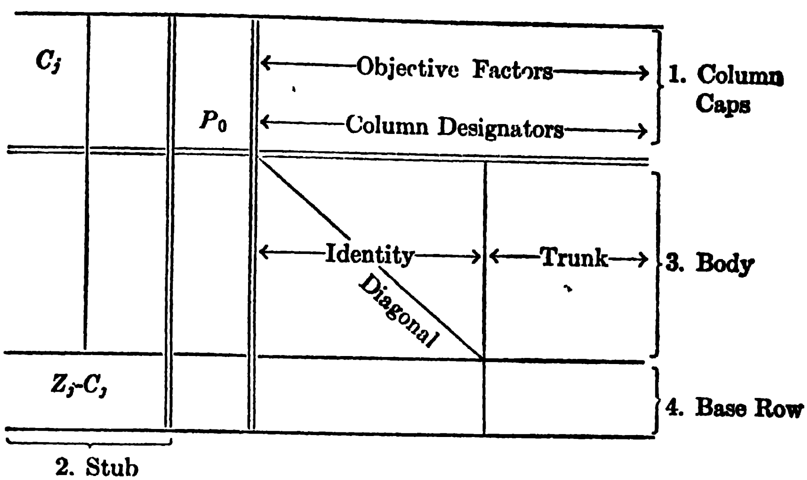


FIG. 4-2. Parts of a Simplex Matrix or Tableau

1. The Column Caps consist of the objective factors and column designators.

2. The Stub consists of the columns to the left of the constant or P_0 column.

3. The Body consists of the columns to the right of the stub and lying between the column caps and the base.

4. The Base Row is the bottom row of the matrix. It serves as an indicator that tells which product to add to the program and when the most profitable program has been reached.

1. Column caps: Each column of a matrix consists of all the coefficients of one unknown. For example, X represents an unknown in the sample problem. The unknown is usually a number of pieces, pounds, yards, or some other unit which we are trying to determine. The cap of the column contains the unit profit or objective coefficient of the unknown.

The equation containing all these caps is called the Objective Equation. (In certain literature on programming these terms are called C's. The C for any column j is referred to as a Cj.) The column cap is made up of

C_j			0	0	0	6	4
		P_0	P_3	P_4	P_5	P_1	P_2
0	P_3	48	1	0	0	2	4
0	P_4	60	0	1	0	4	2
0	P_5	36	0	0	1	3	0
Z_j-C_j							

Column Caps; Body; Base Row

Stub; Identity; Trunk

FIG. 4-3. Simplex Matrix

two parts—the Objective Factors, or Coefficients, and the Column Designators.

a. Objective factors: The Objective Factors are the coefficients of the column designators, or unknown variables in the objective equation. For example, the number 6 is the objective factor for the column designator P_1. The objective factors are expressed as profits per unit, cost per unit, time per unit, and the like.

b. Column designators: The Column Designators are the designating symbols or letters at the head of each column in the body of the matrix—the unknowns or variables. These symbols or letters are letters of the alphabet, such as X, Y, and the like, or they may be one letter with numbered subscripts such as P_0, P_1, P_2, P_3, and the like. The letter P will be used in the ensuing discussion.

2. Stub: The Stub of the matrix is a part of the column caps of the matrix placed in columns at the left-hand side of the matrix. The stub in the first matrix consists of the column caps of the identity. They are arranged in the same vertical sequence as the horizontal sequence to the right in the identity. One entry in the stub changes with each iteration of the matrix.

3. Body: The Body of the matrix consists of the constants of the equations and the coefficients of the unknowns that set up the restrictions or conditions that limit the maximization or minimization. The body consists of three sections, the Constant Column, designated by the letter P_0, the Trunk, and the Identity.

a. Constant column: The Constant Column contains the constants for each equation. It is next to the stub. In some cases it is referred to as the 0 (zero) Column. In technical literature it is referred to as the P_0 Column.

b. Identity: The Identity consists of the coefficients (ones) of the unknowns that are added to the inequations to make them equations. The identity consists of a one and a number of zeros for each equation in the first matrix. There are three inequations in the illustrative product-mix problem—one for each machine center—so that the identity consists of three ones arranged in diagonal stair-step fashion as shown in the simplex matrix. The ones are arranged in a diagonal row running from the upper-left-hand corner to the lower-right-hand corner of the identity. The identity must always be a square. That is, there must be as many rows as there are columns shown in the identity in order to be certain that the computations will work. This is a mathematical requirement, but it is obviously true if all unknowns in the identity are in the stub. The zero profit values of the identity provide the first zero profit program.

c. Trunk: The Trunk of the matrix consists of the original coefficients of the unknowns when the equations were in the form of inequations.

4. Base row: The Base Row of the matrix is the "lowest" row of figures in the matrix. The base row is sometimes called the Index Row. This row contains the "margins" that indicate when improvement can be made and when the best answer has been reached.

ARRANGEMENT OF THE FIRST MATRIX

Before problem information can be put up in a matrix, several steps have to be taken to express the data and information in usable form. These steps include establishing comparable units of measure, converting inequations to equations, and rearranging the equation terms.

1. Establishing comparable units of measure

In an earlier section, it was shown that the modi method requires a standard or common unit of measure into which all data and information are converted. In the simplex method the unit of measure requirements are not as rigid, but they are just as important to the success of the solution. The simplex method requires that units are used that will permit a logical relationship of problem information and answers. These units need not be *identical* for all terms, but they must be *comparable*. For example, in the equation of machine capacity $2X + 4Y + P_8 = 48$, the sum of the hours per piece times pieces for each product adds up to hours—the unit on the other side of the equation. On the other hand, if the production time were expressed as pieces per hour, the units on one side would not be logical in terms of the other. The simplex method permits including several types of restrictions provided that in the computational process the units are comparable. For example, both sales and capacity restrictions can be included in the same matrix when

sales restrictions are expressed in *pieces* and capacity in *hours*. In the computational process the capacity will be used to produce *pieces*, which compares to the *pieces* sold.

2. Converting inequations to equations

An *inequation* is an expression that states certain terms are equal to or less than ($\leqq$) or equal to or greater than ($\geqq$) some value. In order to carry out the simplex calculations, it is necessary to convert all inequalities to equivalent equalities. This is done by adding another term (or terms) to the inequality, thereby making it a usable equality. For example, the expression of the hours available on Machine Center II is $2X + 4Y \leqq 48$. This expression states the time allocated to X and Y on the machine center can be equal to or less than the 48 hours available. This also expresses the way capacity would be regarded in industry. The simplex, however, requires the capacity to be expressed as an equation. This is accomplished simply by adding another term to the expression which has a rate of production of one and a profit of zero. The term can be regarded as if it were another product (an artificial product) which will have a zero value when all capacity is used to produce X and Y or represent idle time when some capacity is not used. The capacity equation with these changes becomes $2X + 4Y + 1P = 48$. The concept of idle time, or slack, is important industrially and mathematically. Industrially, idle time does occur, and it is usually desirable to minimize the effect of it where possible. The mathematics of LP not only provide a way of dealing with slack but also require it to be introduced into the problem in order to use the computational methods. Another rule that applies to converting inequations to equations concerns the direction of the inequality sign. The inequality signs are equal to or greater than ($\geqq$) and equal to or less than ($\leqq$). The rule is that all inequalities must be pointed in the same direction before they are converted to equations. The direction of an inequality can be changed by multiplying or dividing both sides by minus one (-1).

3. Rearranging equation terms

The formation of the first matrix is preceded by a rearrangement of equation terms. The rearrangement is carried out as follows:

1. The constant for each equation is written on the left-hand side of the equation. We take the equation $2X + 4Y + 1P = 48$. In Table 4-2 the number 48 is the constant and is placed on the left. The unknowns and their coefficients are then on the right, except that the idle-time, or slack, variables used to change the inequations to equivalent equations are listed before the other variables and their coefficients. This sequence is the reverse of the way in which equations are usually written, but it does not change the value or relationship. Any unknown that occurs in

one equation must appear in all equations. The unknowns that have no influence on an equation are written in with a zero coefficient as indicated. The equation when rearranged is as follows: $48 = 1P_3 + 0P_4 + 0P_5 + 2P_1 + 4P_2$. This is identical in value and relationship to the equation $2X + 4Y + 1P_3 = 48$.

All equations are rearranged in this fashion and the coefficients of like terms placed in the same column.

Now that the arrangement of equations has been completed and the coefficients and constants placed in the first matrix, the next steps include adding the coefficients of the objective equation and completing the stub and base row.

2. The objective has been established as highest profit (most profitable product mix). In order to maximize profits, a profit per piece is required. The profit per piece for X is \$6, for Y \$4. By use of these values the profit expression, or objective equation, then becomes

$$\text{Maximize: } Z = 6X + 4Y.$$

The coefficients of this expression are written as the caps to their appropriate columns in the matrix. When filled in completely, the cap appears as shown in Figure 4-4.

0	0	0	6	4
P_3	P_4	P_5	$P_1(X)$	$P_2(Y)$

FIG 4-4 Column Caps

There are no profits on the "idle time" or slack terms, so P_3, P_4, and P_5 are assigned a value of zero

3. The stub is the next part of the matrix to be filled in. Stub values are obtained from column-cap values. The objective factors and column designators of the identity columns are listed in rows in the stub. These

Stub

C_j		P_0	0 P_3	0 P_4	0 P_5	Column Caps
0	P_3	48	1	0	0	
0	P_4	60	0	1	0	
0	P_5	36	0	0	1	

FIG. 4-5. Arrangement of Stub

values are placed vertically downward in the stub in the same order that they are listed from left to right in the cap.

When the stub is filled in the matrix for the most profitable product mix, it is as shown in Figure 4-5.

4. *The base, or index, row is all that remains to be filled in to complete the first matrix.* The base is filled in by calculating a value for each column in the base, or index, row to the right of the stub, using the other values in the matrix, which is a computational routine.

COMPUTATIONAL ROUTINE OF THE SIMPLEX METHOD

The computational routine starts by calculating values for the base row of the first matrix.

1. Calculating the base row

Each base- or index-row value is the difference computed by taking the sum of the products formed by multiplying each entry in each column by the corresponding value in the stub, minus the objective, or column cap, for each column in the first matrix. The procedure for calculating the base-row values to complete the first matrix is as follows:

Procedure | Result

1. Select any column to the right of the Stub. Start with Column P_0.

1. Stub

C Profit margin per piece	Product	0 P_0	0 P_3	
				Caps
0	P_3	48	1	
0	P_4	60	0	Body
0	P_5	36	0	
Z_j-C_j		0	0	Base Row evaluated

2. Multiply each entry in the P_0 Column by its corresponding entry in the c (profit per piece) Column and total. This is the profit, or Z, for the program.

2 $0 \times 48 = 0$
$0 \times 60 = 0$
$0 \times 36 = 0$
Total $= 0$

3. Subtract the Cap (C) value for the column and place the difference in the Base Row in the P_0 Column. This is $Z_j - C_j$.

-0
Difference $= 0$

The remaining values in the base row are calculated by the same procedure and are as follows:

0	0	0	6	4
P_3	P_4	P_5	$P_1(X)$	$P_2(Y)$
$0 \times 1 = 0$	$0 \times 0 = 0$	$0 \times 0 = 0$	$0 \times 2 = 0$	$0 \times 4 = 0$
$0 \times 0 = 0$	$0 \times 1 = 0$	$0 \times 0 = 0$	$0 \times 4 = 0$	$0 \times 2 = 0$
$0 \times 0 = 0$	$0 \times 0 = 0$	$0 \times 1 = 0$	$0 \times 3 = 0$	$0 \times 0 = 0$
Total 0	Total 0	Total 0	Total 0	Total 0
−0	−0	−0	−6	−4
Difference 0	Difference 0	Difference 0	Difference −6	Difference −4

Although it is not essential, it will simplify and standardize notation if at this point P_1 is substituted for X and P_2 for Y. When all values have been properly entered, the completed Matrix 1 is as given in Fig. 4-6.

Profit margin per piece	Program products	P_0	0 P_3	0 P_4	0 P_5	6 P_1	4 P_2	Caps
0	P_3	48	1	0	0	2	4	Body
0	P_4	60	0	1	0	4	2	
0	P_5	36	0	0	1	3	0	
Profit margin (in dollars)		0	0	0	0	−6	−4	Base Row

Stub

Iteration 1

FIG. 4-6. Matrix 1 (Tableau 1)

This is the same matrix as the Matrix 1 (Iteration 1) of Table 4-2. The profit of the First Program is zero—indicated by the zero in the P_0 Column. The products that are contained in the Matrix 1, or Tableau 1, are listed in the Program Products Column. These are the idle-time, or slack, products. Each of these has a profit of zero, which of course makes the total program profit zero. The presence of negative values in the Base Row indicates that improvement is possible. Negative six (−6) means that \$6 can be added to the profit of the program for each unit of Product A brought into the program. The addition of each unit of Product B will add \$4 to profits. In summary, then, the first

matrix in a series of matrices is an orderly arrangement of the coefficients of the unknowns and the constants in a family of equations (the other matrices in a series are developed by performing specific computations on the components of the preceding matrix). The parts of a matrix are the objective factors, column designators, stub, body identity, identity diagonal, body trunk, base row, and P_0 column.

The family of inequations and equations used to build up a matrix must have certain qualifications. They consist of an objective and a series of restrictions or limitations on the amount of the objective that can be attained. It must be possible to express the objective and the limitations or restrictions algebraically in linear, or first-degree, equations. All the unknowns that occur in one inequation or equation must be present in all equations. This may mean that many coefficients of the unknowns will be zero. In most cases there are numerous solutions to a system of inequations and equations. The problem is to find the most advantageous feasible solution. By starting with *a* solution (first solution), the simplex technique for manipulating matrices arrives at a best solution by proceeding through successive matrices, using a repetitive or iterative procedure until the most advantageous feasible solution is obtained.

2. Computing successive matrices: the Iterative Process

The solution procedure, starting with a completed first matrix, is to manipulate the values in the first matrix according to an iterative process so that a second matrix is obtained. An *iterative process* is an arithmetic process that repeats itself over and over again, following a standard pattern. Successive matrices are developed according to a pattern until the best solution and program is reached in the last matrix. The complete solution, involving three iterations, is shown in Table 4-2. The number of matrices required to solve a problem will vary with the type and size of problem being solved.

The logical first step to take to obtain a more profitable solution is to determine which product will add the most to the profit per unit and then include it in the next matrix. In this problem that product is A —with a unit profit margin of $6.

Whenever a product is brought into the solution, two conditions have to be considered in order to make sure that a more profitable solution will be obtained. They are:

1. The amount of profit per unit that will be added to the profit total by including the new product in the program

2. The effect of using the time required to produce that product, since manufacturing capacity will be taken away from other products which can add to profits also

Table 4-2. **Most Profitable Product Mix** *

Profit margin per piece (in dollars)	Program		0	0	0	6	4		
		P_0	P_3	P_4	P_5	P_1	P_2		**Ratios**
0	P_3	48	1	0	0	2	4	Iteration 1	$^{48}/_2 = 24$
0	P_4	60	0	1	0	4	2		$^{60}/_4 = 15$
← 0	P_5	36	0	0	1	3	0		$^{36}/_3 = 12$ ✓
		0	0	0	0	−6	−4		
← 0	P_3	24	1	0	−⅔	0	4	Iteration 2	$^{24}/_4 = 6$ ✓
0	P_4	12	0	0	−⅘	0	2		$^{12}/_2 = 6$
→ 6	P_1	12	0	0	⅓	1	0		$^{12}/_0 = \infty$
		72	0	0	2	0	-4		
→ 4	P_2	6	¼	0	−⅙	0	1	Iteration 3	
0	P_4	0	−½	1	−1	0	0		
6	P_1	12	0	0	⅓	1	0		
Highest profit margin (in dollars) →		96	0	2	⅘	0	0		

Most Profitable Product Mix *

12 pieces of Product A at \$6 per piece = \$72
6 pieces of Product B at \$4 per piece = 24

Total \$96

Use of Machine Time

	A		B	
Machine Center I	2 hours per piece × 12 pieces	+	4 hours per piece × 6 pieces	= 48 hours
Machine Center II	4 × 12	+	2 × 6	= 60 hours
Machine Center III	3 × 12			= 36 hours

* Mix that yields the highest manufacturing margin.

This can be seen more clearly by considering what happens when one piece of Product A is brought into the solution. It will add its unit profit of $6. But adding one piece of Product A will require production *time*, and that production time must be taken away from the time that could be allocated to other products, since there is only a limited amount of production time available. Based on the first solution, producing one piece of Product A (P_1) will require 2 hours on Machine I, which will take away productive capacity or time for the production of other products on that machine—including idle time (P_3) and Product B (P_2).

3. Manipulating matrix values

Successive matrices are computed from values in the preceding matrix. The procedure for developing a new matrix requires first the selection of a key column (other than the P_0 Column) and a key row in the current matrix. *By definition a Key Column is that column which has the largest or most negative number in the base or index row. It is the column whose product will add the most profit per unit.*

Looking at Matrix 1 which has been computed, we find that the largest negative number in the Base, or Index, Row (aside from the P_0 Column) is -6. This designates the P_1 Column as the Key Column. Generally, the key column is "boxed" so as to set it off for ease of calculation. The key column is usually a different column in successive matrices. The final matrix does not require a key column because it is not necessary to develop another matrix.

The key row is also found for each matrix except the last one. The key row is always located in the body of the matrix. It is selected by dividing each *positive* number in the key column into a positive number or *zero* in the same row in the constant column and selecting the row with the *smallest* ratio. *This smallest ratio represents the bottleneck or factor which limits the amount of profit.* Referring to Matrix 1, we take the positive values in the Key Column, which are 2, 4, and 3, and divide them into the corresponding Row values in the P_0 Column, which are 48, 60, and 36. The smallest ratio—$36/3 = 12$—is obtained by using the third row. This is the Key Row. It is boxed to make subsequent calculations easier to follow and to indicate that it is the first row that is to be placed in the matrix being developed.

In some cases, two ratios may turn out to have the same value, in which case a tie exists. For example, the ratios might be $24 \div 4$ and $12 \div 2$, which are the same in value. When this occurs, the row which has the larger value in the key column, in this case row 4, is selected as the key row. Another case where breaking a tie is necessary before a key row can be selected frequently occurs when there are zeros (0's) appearing

in the P_0, or constant, columns. For example, we might find that 3 and 4 are the only positive values in the key column and that the values in the P_0, or constant, column into which they are to be divided are both zero (0). The ratio of 0/3 and 0/4 are both zero so that a tie exists between the two rows that are to be replaced. In this situation, the tie is broken by dividing the numbers in the key column for the two rows into each value in their respective rows starting at the constant column and moving to the right and comparing at each column. When one ratio becomes smaller than the other, the tie is considered broken and the row containing the smaller value is designated as the key row. When the key row has been selected, it is used to form the Initial Row in the next matrix. *The initial row assumes the same row position in the new matrix that the key row has in the current matrix.* The initial row is so named because it is the first row calculated and entered in the new matrix.

The number in the location at the intersection of the key column and key row is called the Key Number. In Matrix 1, the Key Number is 3 and for ease of reference is circled or boxed. The Initial Row, or first row, of the next matrix is formed by dividing each number in the Key Row (of the current matrix) by the Key Number and placing each value thus obtained in the initial row and the appropriate column of the new matrix, starting with the Constant Column. The values in the initial row of Matrix 2 are 12, 0, 0, ⅓, 1, and 0. The initial-row entries in the stub are made up of the product selected to come in (in this case P_1 replaced P_5) and the profit per piece of that product (in this case 6 replaces 0). Then the remainder of the stub is copied directly from the preceding matrix.

The other entries of the body and base row of the new matrix have to be calculated in order to obtain a solution. When this solution is completed, it will indicate a better program than the previous solution and indicate whether the best program has been calculated or not.

The procedure for completing the matrix is as follows:

1. Select a square in the new matrix to be filled in.

2. Refer to the value occupying the corresponding square in the preceding matrix.

3. Subtract from that value the product formed by the ratio of the number at the intersection of the column containing the value and the key row and the key number and by the number found in the same row as the original value selected but in the key column.

4. Place the result of these computations in the square in the new matrix.

5. Perform the same type of calculations for all remaining squares, including those in the base row, until all squares have been filled in.

By using these rules, the value to be inserted in the square in Matrix 2 at P_8P_8 is computed as follows:

1. Refer to the value occupying the same square in the preceding matrix. This value is 1.

2. Subtract from that value the product formed by the ratio of the number at the intersection of the column containing the value and the Key Row (0) and the Key Number, which is 3 (the ratio is $0/3 = 0$), and multiply that ratio by the number found in the same row as the original value but in the Key Column. This value is 2. The result of steps 1 and 2 is

$$1 - (0/3)\,(2) = 1$$

3. Place the result in the appropriate square of the new matrix.

The steps for determining the values to be filled in the squares of successive matrices can be expressed in the form of equations as follows:

1. Initial row $\dfrac{\text{key row}}{\text{key number}}$

2. All other new entries $= \left(\begin{array}{l}\text{number in}\\ \text{old matrix}\end{array} - \dfrac{\begin{array}{c}\text{corresponding}\\ \text{key-row number}\end{array} \times \begin{array}{c}\text{corresponding key-}\\ \text{column number}\end{array}}{\text{key number}}\right)$

By using this procedure, the most profitable solution is obtained in three matrices (all values in the base row are positive), as indicated in Iteration 3 of Table 4-2.

The most profitable mix and profit can be read directly from the P_0 Column of the last matrix. The mix consists of 12 pieces of Product A (P_1) and 6 pieces of Product B (P_2), which gives a profit of \$96. There is no other program that will yield a higher profit for the conditions being considered than the one calculated.

Frequently it happens that there are alternative *best* programs—that is, different combinations of products that provide the *same* profit. These are indicated by a zero entry in the base row of the final matrix in a column for a product or activity not appearing in the final solution. In the sample problem just shown, there is no alternate best program for the most profitable product mix. The two zero entries in the Base Row of Matrix 3 are in columns whose entries appear in the final solution.

SUMMARY

The rules for computing successive matrices and short cuts for developing a new matrix are summarized as follows.

1. Rules for computing successive matrices

1. Select a key column, key row, and key number.
2. Divide the key number into each value in the key row to the right of the stub and place the ratios in the initial row in the next matrix. The initial row occupies the same row position in the new matrix as the key row occupied in the previous matrix.
3. Form the stub for the initial row. Complete the stub for the new matrix by copying all other entries from the stub of the preceding matrix into their corresponding position in the new matrix.
4. Compute the values for all empty squares that remain in the new matrix by using the formula (start by evaluating the base row):

$$\text{New values} = \left(\begin{matrix}\text{number in} \\ \text{old matrix}\end{matrix} - \frac{\begin{matrix}\text{corresponding} \\ \text{key-row number}\end{matrix} \times \begin{matrix}\text{corresponding key-} \\ \text{column number}\end{matrix}}{\text{key number}}\right)$$

5. Go no further when all numbers in the base row become positive. The best solution has been reached.

There are a number of short cuts and checks that are helpful in developing a new matrix. These are given below.

2. Short cuts and checks for developing a new matrix

1. The base row of each successive matrix can be computed by the formula and by the method used in obtaining values for the base row of the first matrix.
2. The column of numbers under an objective that also appears in the stub is always a number of zeros and a single one.
3. If a zero appears in the key column, all the numbers in this row are the same in the next matrix.
4. Similarly, if a zero appears in the initial row of the new matrix, all the numbers in this column are the same in the next matrix.
5. The base row should be figured next after the initial row of the present matrix. If this *is* the last matrix in the series, there will not be any negative numbers in the base. Then it will only be necessary to figure the P_0 Column in the present matrix to complete the solution.

It is important to examine the significance of some of the calculations and the values to see more clearly what is done in the simplex method.

As was stated earlier, the simplex is a method of solving a set of linear equations and inequations in such a way as to maximize or minimize the value of a linear objective containing the same variables found in the equations and inequations. The final solution will contain values for the variables or unknowns which, when substituted in the objective equation, maximize the amount of the objective.

The first matrix, which is an orderly arrangement of the problem information in rows and columns, represents *a* solution. In other words, the simplex method provides a solution to start with which is a zero profit but feasible solution. This is comparable to the zero point in the geometric solution of the problem. By following an iterative procedure successively better solutions are obtained until the best solution is reached.

The constant, or P_0, column will always contain a program and the result (or profit) to be obtained by following that program. The column will therefore contain values which, when substituted in the objective equation, will give the value shown in the square at the base of the column in the base row. In Matrix 1 of Table 4-2 all the time is idle time (P_3, P_4, and P_5 are by definition idle time). Nothing is manufactured, and the profit is zero. The first three columns to the right of the P_0 Column list the slack or idle-time variable and the next two columns list the real products that can be made. These columns represent the technological framework within which the problem is to be solved—that is, they contain values that represent the productive-process time. The rows below the Cap in Matrix 1 other than the Base Row contain the constants and coefficients of the restrictive equations that limit what can be produced. The top row shows the unit profit that is possible. The Base Row shows the product profit margins that are available by bringing their respective profits into solution. The simplex calculations proceed to use up the slack, or idle time to produce something at a profit. As long as there are negative values in the base row, profits can be increased.

The row selected to be replaced represents a bottleneck. It is selected on the basis of the smallest ratio, because it represents the largest number of complete items that can be produced at that point in the computations.

In Matrix 2, the values in the P_0 Column are significant. *The values appearing in the Initial Row represent the number of pieces of the product that can be produced at the bottleneck. All other values represent the remaining hours that are available for production of other products.* The figure at the base of the P_0 Column is the profit that the program will give. To refer again to the geometric solution, the computations of Matrix 2 show that the solution has moved from zero (0, 0) to $X = 12$, $Y = 0$ (12, 0) and a profit of \$72. The row to be replaced in the next matrix is again a bottleneck row. The Base Row in Matrix 2 contains a negative value, which indicates that a better program is possible.

In Matrix 3, Column P_0 again contains the program to be followed to obtain the results that the computations indicate are possible. This can also be seen in the geometric solution at $X = 12$, $Y = 6$. In this case,

Matrix 3 is the final matrix so that the P_0 Column contains the best program. The most profit is obtained by manufacturing 12 pieces of A and 6 pieces of B. The figure at the base of the column is the maximum profit possible under the conditions of the problem. The Base Row contains only positive values, which indicate that the best solution has been reached. If an item having a positive value is added to the program in a greater quantity than at present, a decrease in profit will result.

In summary, the simplex method provides:

1. A precise way of approaching or stating a problem where a number of solutions are possible
2. A method which analyzes the problem data and information until a best solution has been obtained
3. A plan or program for obtaining the best program or any of the less profitable programs
4. A basis for management decision based on the program and alternates provided
5. A method for reevaluating the plan or program as conditions and information change

CHAPTER 5

The Simplex Method: Part 2

VARIOUS TYPES OF BEST SOLUTIONS USING THE SIMPLEX METHOD

The simplex method is not confined solely to solving problems of most profitable product mix. It can be used to provide a number of different best answers and the programs associated with each. For example, *different* objectives, such as least cost, least time, least distance moved, and the like, can be solved for and information provided management relative to the particular objective being considered.

In addition to solving for different objectives, the simplex can be used to explore the desirability of following a variation from a *particular objective.* For example, management may want to know the most profitable program for each of several different levels of sales commitments, highest material turnover, and the like.

Since the basic computational steps are the same irrespective of the objective, it is necessary to set up the problem in such a way that the method will provide the desired answer. This generally involves including a particular condition or restriction which can be taken into account as the calculations proceed. The method will not provide an answer that reflects a given condition, unless provisions are made or steps taken to include the conditions or restrictions when one is formulating and setting up the problem.

The principal purpose of this chapter is to demonstrate the reasoning and steps involved in solving for various objectives and answers—particularly from the standpoint of the method itself. In many cases it has been helpful to ask the question: "What does the method have to do to take account of this particular condition when it is imposed on the problem and what will it do about it as the calculations proceed?"

Another important but secondary purpose is to evaluate the answers and their significance. The answer is the final test of how well the problem was formulated in the first place. The final answer and program also provide valuable information for management consideration.

To demonstrate the usefulness and versatility of the simplex method

a number of problems will be discussed and solved. These are grouped into two general classes for purposes of presentation. One group involves solving for objectives different from most profit, for example, objectives that involve minimizing time or cost. The second class involves solving for most profit considering a number of variations and conditions. The computational process is the same in both cases, but there are differences in the formulation of the problems which must be considered in detail.

SOLVING FOR VARIATIONS OF HIGHEST PROFIT MARGIN: THE DRAWROD COMPANY

The various steps involved in evaluating variations to a particular objective from the standpoint of *method* and *result* can best be demonstrated by an actual problem. For this purpose, information is provided about the Drawrod Company, whose management wants to investigate the gross-profit margin under different conditions of operation.

THE DRAWROD COMPANY

The Drawrod Company, a hypothetical manufacturing firm, manufactures drill rod from coils of wire. The drill rod is supplied to customers, who in turn use it to make drills and parts for power tools, dowel pins, and similar metal parts.

The rod is made from mill wire, which is purchased in coils of one standard size and weight from a nearby basic producer. The coils as they come from the mill have a wire diameter, or outside diameter (OD), of .5313 ($^{17}/_{32}$) inch and weigh approximately 160 pounds. Manufacturing methods and material-handling equipment have been set up to handle and convert the standard coil to rods 12 feet long. Rods are produced in .5000 OD, .3750 OD, and .2500 OD.

A block diagram of the manufacturing process used to reduce the mill size to customer size is given in Figure 5-1.

1. Steps in the manufacturing process

The first step in the process is to anneal the coils when they are received from the mill. The coils are placed on edge in rows in the annealing furnaces. After Anneal, the coils are pickled to remove the scale prior to drawing.

The second step is to reduce the wire diameter to desired size by means of draw blocks. The draw blocks consist of a spindle on which the coiled wire is loaded and a spindle on which the drawn wire is coiled as it is drawn through the die block. There are limitations to the amount of reduction that can be made in cross section at each draw because of the stresses that are set up in the material by the drawing process. The

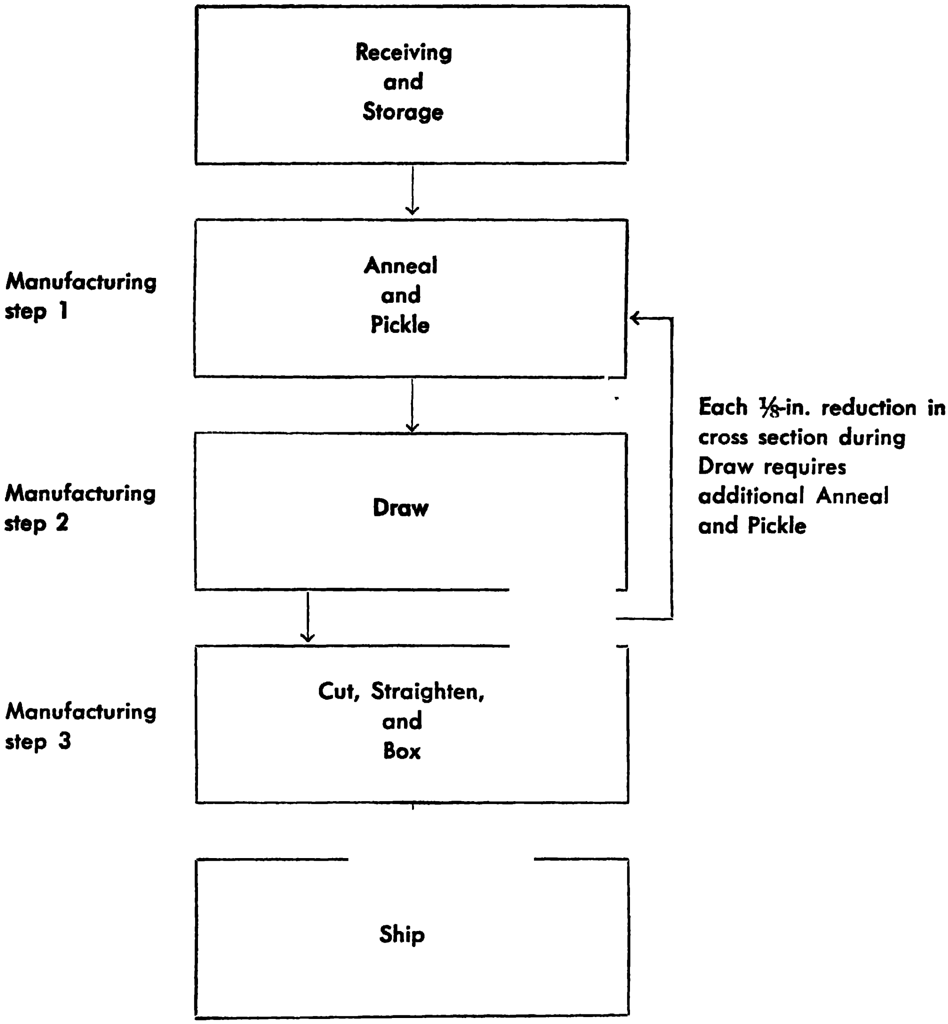


FIG. 5-1. General Manufacturing Process
The Drawrod Company

general rule that is followed is that each ⅛-inch reduction in cross section requires an Anneal to relieve the stresses. This means that for smaller diameters there will be a number of annealing and pickling operations before a finished size is obtained. As the coil is reduced in diameter, it elongates so that for some sizes the coil is cut in half for ease of handling. Otherwise, it becomes too bulky to handle.

The third step in the process is to Cut to Length, Straighten, and Box. After the coil is drawn to finished size on the draw blocks, it is then sent to a cutter and straightener, which straightens the wire and cuts it into rods 12 feet long. The runout table is set up so that the shipping boxes can be loaded automatically by tilting a section of the runout table. The

boxes are then placed on four-wheel wagons (or buggies) for movement to the Shipping Department.

Specific production and capacity information relating to finished size, margins, operations, utilization, and available hours is given in Table 5-1.

Table 5-1.

A. Production and Margin Information

Finished size (in inches)	Operation	No.	Hours per 100 lb	Manufacturing Margin per 100 lb * (in dollars)
.5000	Anneal and Pickle	1	.10	2.10
	Draw	1	.05	
	Cut, Straighten, and Box	1	.07	
.3750	Anneal and Pickle	2	.23	4.80
	Draw	2	.13	
	Cut, Straighten, and Box	1	.09	
.2500	Anneal and Pickle	3	.26	6.00
	Draw	3	.22	
	Cut, Straighten, and Box	1	.11	

B. Departmental Capacities

Operation	No. machines	Shifts per week	Hours of operation	Per cent utilization †	Available hours per week
Anneal	5	20	800	.875	700
Draw	8	10	640	.859	550
Cut, Straighten, and Box	6	10	480	.937	450

* Obtained from Accounting Department. Does not include fixed costs. Does include labor and burden where it can be assigned.

† Adjusted for down time, including maintenance, breakdowns, and absenteeism.

Using this information, management wants to investigate the gross-profit margin under different conditions as a basis for profit planning. They are interested in developing answers and programs for highest manufacturing margin—without sales restrictions, least idle time—maximum utilization of equipment, highest material turnover—largest number of pounds, highest manufacturing margin—meeting *exact* sales requirements, and highest manufacturing margin—meeting *minimum* sales requirements.

2. Basic assumptions

Linear-programming solutions and the programs associated with them assume a fixed and specific period of time. The solution presented for each planning period is independent of the solution for any other period. For this reason, linear programming tends to be more useful as a planning tool in the short run because some of the data assumed to be fixed for each problem solution may change over a long-time period and thus render the solution invalid. This can be overcome by computing new solutions for successive time periods of short duration and by computing new solutions as new data become available.

In the profit-planning analysis for the Drawrod Company, each solution assumes that the available capacity of the various departments is fixed and limited for the time period being considered. Raw materials and labor are available at constant prices, and the rates of production are assumed to remain unchanged. No allowances for changes in the prices of finished products are permitted. It is also assumed that for short periods "going charges," such as taxes, insurance, and supervision, are fixed.

3. Highest manufacturing margin—without sales restrictions

The program of highest manufacturing margin—without sales restrictions—provides useful information for evaluating other programs that production management may want to consider. The highest-margin program without regard to sales provides an insight into the optimum product mix and margin that can be obtained, considering the operations, facilities, products, and other conditions of the problem. Although it may not be a realistic program in actual practice, it does provide a yardstick and information that management can consider in its planning and decision making.

Setting up the problem

The three products that can be produced are designated as $P_1 = .5000$ OD, $P_2 = .3750$ OD, and $P_3 = .2500$ OD. The economic choice to be

made is the selection of the amounts of each of the products to be produced for the time period (1 week in this problem) subject to the restriction that no more than 100 per cent of the available hours of any department can be used.

Each product requires processing in each one of the three departments in order to obtain a complete or finished product. Only finished products will be admitted in the final solution, and each will compete with the other for use of capacity. The capacity is to be used in such a way that the highest manufacturing margin results. This objective expressed as an equation is

$$\text{Highest manufacturing margin} = \$2.10P_1 + \$4.80P_2 + \$6.00P_3$$

In this expression, the values \$2.10, \$4.80, and \$6.00 represent the respective manufacturing margins per unit for each of the three products. The amounts of P_1, P_2, and P_3 that maximize manufacturing margin are to be determined.

The restriction on the amount of manufacturing margin, given the rates of production, is the amount of capacity that is available. Anneal and Pickle has available for production of the three products a total of 700 hours for the week. This restriction, expressed in terms of the products that compete for the use of the available time, is $700 \geqq 10P_1 + 23P_2 + 26P_3$. This expression states that the three products can use any amount up to and including all the 700 hours available. The values 10, 23, and 26 are the hours per unit, respectively, for each of the three products. Similarly, the expression of Draw capacity in terms of the products that compete for use of the available time is $550 \geqq 5P_1 + 13P_2 + 22P_3$. Likewise, the expression of Cut, Straighten, and Box capacity in terms of the products is $450 \geqq 7P_1 + 9P_2 + 11P_3$. Since it is not possible to determine in advance whether it is more profitable to allow some of the available capacity to remain unused during the week, a mathematical step must be introduced to permit this possibility to be considered in the solution. This is done by introducing disposal, or idle-time, products, which permit the method to determine whether some of the departmental capacities should remain idle. The disposal, or idle-time, products (sometimes referred to as Dummy Products) are designated as P_4 for Anneal and Pickle, P_5 for Draw, and P_6 for Cut, Straighten, and Box.

By assuming an arbitrary rate of production of one hour per unit and a profit of zero for each dummy product, we can prepare Table 5-2, which collects and summarizes the forgoing information related to setting up the problem for solution.

This orderly arrangement of problem information with the addition

Table 5-2. Data Arrangement for Solution by the Simplex Method

Departments	Hours available	Products					
		Disposal (idle time)			Actual		
		P_4	P_5	P_6	P_1	P_2	P_3
Anneal	700	1			10	23	26
Draw	550		1		5	13	22
Cut, Straighten, and Box	450			1	7	9	11
Manufacturing margin (in dollars)		0	0	0	2 10	4.80	6 00

of a few changes required by the method readily becomes Matrix 1 (Tableau 1) of Table 5-3.

Tableau 1 shows by the entries in the Program Column that the products that make up the first program are the disposal, or idle-time, products P_4, P_5, and P_6. This means that at the starting point in planning production using the simplex method, all the departmental capacities are assumed to be unused or idle. When this is the case, the manufacturing margin is assumed to be zero (no profit without the production of actual products). From this beginning or starting point (Tableau 1), the simplex method makes it possible to compute successively higher margin programs in separate programs or tableaus until the highest margin program and profit is calculated.

Tableau 2 is developed from the data provided in Tableau 1. Basically, the procedure of developing one tableau from another is one of selecting one product to bring into the program that will add the most to the manufacturing margin. In Tableau 1, this is P_3, which has a unit margin of \$6. This is indicated by the figure -6.00 shown in the base row. The -6.00 means that a unit margin of \$6 will be forgone for each unit of P_3 not included in the next tableau.

Once it is decided that P_3 is to be made because it offers the largest profit opportunity, the next decision is to determine how much of it is to be made in the next program. This is established by dividing the time

Table 5-3. Highest Manufacturing Margin—without Sales Restrictions Drawrod Company

Margin per unit (in dollars)	Program	P_0	0 P_4	0 P_5	0 P_6	2.10 P_1	4.80 P_2	6.00 P_3		Ratios
					Tableau 1					
0	P_4	700	1	0	0	10	23	26	Iteration 1	700/26 = 26.923
← 0	P_5	550	0	1	0	5	13	22		550/22 = 25.000 ✓
0	P_6	450	0	0	1	7	9	11		450/11 = 40.909
		0	0	0	0	−2.10	−4.80	−6.00		
					Tableau 2					
← 0	P_4	50.00	1	−1.182	0	4.098	7.634	0	Iteration 2	50/7.634 = 6.549 ✓
→ 6.00	P_3	25.00	0	.045	0	.227	.591	1		25/.591 = 42.301
0	P_6	175.00	0	−.499	1	4.503	2.499	0		175/2.499 = 70.028
		150.00	0	.273	0	−.738	−1.254	0		
					Tableau 3					
⇄ 4.80	P_2	6.549	.131	−.155	0	.537	1	0	Iteration 3	6.549/.537 = 12.200 ✓
6.00	P_3	21.130	−.077	.137	0	−.090	0	1		
0	P_6	158.634	−.327	−.113	1	3.162	0	0		158.634/3.162 = 50.175
		158.21	.164	.079	0	−0.65	0	0		
					Tableau 4					
→ 2.10	P_1	12.20	.244	−.288	0	1	1.863	0	Iteration 4	
6.00	P_3	22.23	−.055	.111	0	0	.168	1		
0	P_6	120.06	−1.099	.799	1	0	−5.889	0		
Highest profit (in dollars)		159.00	.180	.060	0	0	.121	0		

Highest Manufacturing Margin and Program—without Sales Restrictions
(Expressed in 100-pound units)

12 2 units, or 1,220 lb	Product 1 (P_1 = 5000 OD)	at $2 10	=	$ 25 62
0 units, or 0 lb	Product 2 (P_2 = 3750 OD)	at 4 80	=	0
22 23 units, or 2,223 lb	Product 3 (P_3 = 2500 OD)	at 6 00	=	133 38
Total				$159 00

Departmental Load Hours

	P_1	P_2	P_3	Load hours	Available hours	Unused hours
Anneal	122	0	578	700	700	0
Draw	61	0	489	550	550	0
Cut	85 4	0	244 6	330	450	120

available in the three departments shown in the P_0 Column by the time per piece. The division shows that the Draw Department is the bottleneck because it will permit only 25 units of P_3 to be made (550 hours available ÷ 22 hours per unit = 25 units). This computation for all departments is shown under the Ratios Column. The fact that the other departments permit a larger number of units of P_3 to be manufactured is of no consequence at this point because only 25 *completed* units can be processed through Draw, which is the limiting department.

Successive tableaus are computed, one from the other, by the simplex method until the optimum program is reached in Tableau 4. The Base Row of Tableau 4 does not contain any negative values, which means that the program shown in Tableau 4 has not overlooked any margin possibilities for increasing profit and that the highest-margin program has been calculated.

The highest manufacturing margin program without regard to sales restrictions consists of 12.2 units of P_1 (1,220 pounds of .5000 OD), 22.23 units of P_3 (2,223 pounds of .2500 OD) and zero units of P_2 (0 pounds of .3750 OD). The highest margin is indicated at the base of the P_0 Column as $159.00. Any program other than this—especially one that attempts to employ the unused time in the Cut, Straighten, and Box Department—will result in a lower margin.

The values in the Base Row of Tableau 4 offer useful by-product information for production-management planning. For example, the value .180 shown at the base of Column P_4 indicates that if 1 hour's capacity is added to Anneal, the manufacturing margin can be increased by $.18. Under such a program the product mix changes to 1,244 pounds of P_1 and 2,217 pounds of P_3. P_2 is not included in the solution.

Evaluating the answer

The most-profitable-product-mix answer indicates the production program that should be followed to maximize manufacturing margin—provided the firm can sell everything that it produces. This may not be very practical because of commitments to customers, but the profit and program do provide a useful goal and target for production-management planning. The useful information that the solution provides management is as follows:

1. The highest manufacturing margin potential of the plant in terms of the products, rates of production, capacity, and rate of profit is $159.00 for the period.
2. In the highest-margin program there are 120 hours of unused time on the cutting and straightening operation. The production bottlenecks are the annealing and drawing operations.
3. Product 2 (.3750 OD) is not included in the most profitable program despite a unit profit of $4.80 per unit. The calculations indicate that the time required to produce $4.80 profit margin making Product 2 can be more profitably used to produce other products, *all products considered.*
4. A departmental load can be calculated for scheduling and manning purposes, and is shown in Table 5-3.

4. Least idle time—maximum utilization of equipment

Frequently, management plans production to make maximum use of equipment. Often this would seem to be a good policy for profitable operation—particularly when management is accustomed to thinking of plant-earning capacity on a per machine-hour basis. For this reason, the fact that the most profitable program includes 120 hours of idle time on one machine center may appear to be questionable. Should it not be possible to make additional profit by making some use of this idle capacity? Table 5-4 shows how this situation may be explored.

Setting up the problem

Again the capacity limitations are the same, but this time the objective is to maximize utilization or minimize idle time. For this objective, each slack variable is associated with a unit penalty. In other words, each unit of slack represents an hour of idle time. The ones are made negative since the objective is to minimize.

The product columns are all capped with zeros, since no (or zero) idle time is associated with their production. This also compels the method to consider the three to be equally desirable from the standpoint of machine loading. Relative profitability is ignored. When the problem

Table 5-4. Highest Manufacturing Margin—Least Idle Time (Maximum Utilization of Equipment)

Drawrod Company

Margin per unit (in dollars)	Program	P_0	P_4	P_5	P_6	P_1	P_2	P_3		Ratios
			−1	−1	−1	0	0	0		
						Tableau 1				
−1	P_4	700	1	0	0	10	23	26	Iteration 1	700/26 = 26.923
← −1	P_5	550	0	1	0	5	13	22		550/22 = 25.000 ✓
−1	P_6	450	0	0	1	7	9	11		450/11 = 40.909
		−1700	0	0	0	−22	−45	−59		
						Tableau 2				
← −1	P_4	50.00	1	−1.182	0	4.098	7.634	0	Iteration 2	50/7.634 = 6.549 ✓
→ 0	P_3	25.00	0	.045	0	.227	.591	1		25/.591 = 42.301
−1	P_6	175.00	0	−.499	1	4.503	2.499	0		175/2.499 = 70.028
		−225	0	2.681	0	−8.601	−10.131	0		
						Tableau 3				
⇆ 0	P_2	6.549	.131	−.155	0	.537	1	0	Iteration 3	6.549/.537 = 12.200 ✓
0	P_3	21.130	−.077	.137	0	−.090	0	1		
−1	P_6	158.634	−.327	−.113	1	3.162	0	0		158.634/3.162 = 50.175
		−158.634	1.327	1.113	0	−3.162	0	0		
						Tableau 4				
0	P_1	12.200	.244	−.288	0	1	1.863	0	Iteration 4	
0	P_3	22.230	−.055	.111	0	0	.168	1		
−1	P_6	120.063	−1 099	.799	1	0	−5.889	0		
Least number idle hours		120.063	2.099	.2011	0	0	5.890	0		

Highest Manufacturing Margin and Program—Least Idle Time (Expressed in 100-pound units)

12.2 units, or 1,220 lb Product 1 (P_1 = .5000 OD) at \$2.10 = \$ 25.62
0 units, or 0 lb Product 2 (P_2 = .3750 OD) at 4.80 = 0
22.23 units, or 2,223 lb Product 3 (P_3 = .2500 OD) at 6.00 = \$133.38

Total \$159.00

Departmental Load Hours

	P_1	P_2	P_3	Load hours	Available hours	Unused hours
Anneal	122	0	578	700	700	0
Draw	61	0	489	550	550	0
Cut	85.4	0	244.6	330	450	120

is set up in this way the method is made to overlook or neglect the profit of each product and provide an answer that minimizes idle time.

Evaluating the answer

It may seem peculiar that in a program of minimum idle time there are still 120 hours of unused time on the Cut and Straighten equipment. This results primarily from the way in which the problem is set up. As far as the method is concerned, it was told to consider only completed products. Each activity or real product required three operations and did not permit partially completed products to enter the calculations. A check of Tableau 1 in Table 5-4 shows that each real-product column has three entries—one for each operation. It is possible to set up the problem to consider partially completed products if desired. A tabulation of the least-idle-time answers for various product schedules and the information they provide have proved useful in industry from the standpoint of rearranging man power and selecting new and additional equipment. The program of least idle time has the same profitability, products, and amount of production as the most-profitable-product-mix solution. The complete simplex solution to the problem is given in Table 5-4. The least number of idle hours is read directly from Matrix 4.

5. Highest material turnover—largest number of pounds

Another program of interest to management is the one which would result in the highest possible turnover of material (maximum tonnage). Such a program can be calculated readily from the information available.

Setting up the problem

The conditions and limitations of capacity are exactly the same as in the most-profitable program. Only the objective is changed. Instead of considering the unit profit that is made on each product, the maximum-material-turnover solution considers the amount of material that is contained in each product. Since 100 pounds of product contain 100 pounds of material, the material-turnover rate for all products is unity.

Table 5-5 shows how the problem is set up and its solution. The matrix is the same as for the most-profit solution except for the Column Caps. The three slack variables (representing idle time) still carry a value of zero—no material turnover. The product columns, however, are capped with a value of 1—indicating 1 pound of material turnover for each pound of product made.

Evaluating the answer

The largest number of pounds are produced when capacity is used to manufacture Product 1 (.5000 OD). It requires only 22 hours per hun-

Table 5-5. Highest Material Turnover—Largest Number of Pounds
Drawrod Company

Margin per unit (in dollars)	Program	P_0	0 P_4	0 P_5	0 P_6	1 P_1	1 P_2	1 P_3		Ratios
Tableau 1										
0	P_4	700	1	0	0	10	23	26	Iteration 1	700/26 = 26.923
← 0	P_5	550	0	1	0	5	13	22		550/22 = 25.000 ✓
0	P_6	450	0	0	1	7	9	11		450/11 = 40.909
		0	0	0	0	−1	−1	−1		
Tableau 2										
← 0	P_4	50.00	1	−1.182	0	4.098	7.634	0	Iteration 2	50/4 098 = 12.200 ✓
→ 1	P_3	25.00	0	.045	0	.227	.591	1		25/.227 = 110.132
0	P_6	175.00	0	−.499	1	4.503	2 499	0		175/4.503 = 38.863
		25.00	0	.045	0	−.773	−.409	0		
Tableau 3										
→ 1	P_1	12.20	.244	−.288	0	1	1 863	0	Iteration 3	
1	P_3	22.230	−.055	.111	0	0	.168	1		22 230/.111 = 200.459
← 0	P_6	120 063	−1.099	.799	1	0	−5 889	0		120.063/.799 = 150.304 ✓
		34 43	.189	−.177	0	0	1.031	0		
Tableau 4										
1	P_1	55.547	− 153	0	361	1	−.263	0	Iteration 4	
← 1	P_3	5.562	.097	0	−.139	0	.986	1		5.562/.986 = 5.643 ✓
→ 0	P_5	150.304	−1.375	1	1 252	0	−7.372	0		
		61.10	−.055	0	.222	0	−.277	0		
Tableau 5										
1	P_1	57.030	−.127	0	.324	1	0	.267	Iteration 5	
⇄ 1	P_2	5.643	.099	0	−.141	0	1	1 015		5.643/.099 = 57.30 ✓
0	P_5	191.094	−.649	1	.214	0	0	7.479		
		62.66	−.028	0	.183	0	0	.281		
Tableau 6										
1	P_1	64.29	0	0	.143	1	1.285	1.572	Iteration 6	
→ 0	P_4	57.29	1	0	−1.429	0	10.152	10.299		
0	P_5	228.60	0	1	−.714	0	6.592	14.166		
Largest number pounds		64.28	0	0	.143	0	.285	.571		

Highest Manufacturing Margin and Program—Greatest Quantity
(Expressed in 100-pound units)

64.28 units, or 6,428 lb Product 1 (P_1 = .5000 OD) at \$2.10 = \$135.00
Departmental Load Hours

	P_1	P_2	P_3	Load hours	Available hours	Unused hours
Anneal	642.9	0	0	642.8	700	57.2
Draw	321.4	0	0	321.4	550	228.0
Cut	450	0	0	450	450	0

dredweight, whereas both Product 2 and Product 3 require considerably more hours to produce the same weight. The answer in this problem tends to be obvious. But in other problems, where many rates of production, machines, capacities, and the like must be considered, it becomes a more difficult problem, and the answer is not easily seen. This is especially true when certain operations—such as drawing—can be carried out on draw blocks having different rates of production.

Experience to date indicates that highest-tonnage solutions generally come as a surprise to management when compared to most profitable programs. The amount of difference in profit obtainable from the same machines in the same *time* under the two different programs has been somewhat startling. The comparison of the highest margin and highest material programs for the Drawrod Company gives an indication of this:

	Profit	Pounds	Unused hours	Per cent profit increase
Highest poundage	\$135	6,428	120.0	—
Highest margin	159	3,443	285.8	17.8

SOLVING FOR VARIATIONS TO HIGHEST MANUFACTURING MARGIN

Another class of answers that is useful to management planning and decision making includes variations of highest manufacturing margin such as those introduced when sales restrictions are imposed on production facilities. A number of successful applications [1] indicates that infor-

mation regarding the effect of various sales commitments on production and profit provided by linear-programming methods has proved useful in arranging the product mix to be scheduled in a period. In addition, the sales department is provided with information about the profit-earning potential of the product lines and the status of customers' orders.

1. Highest manufacturing margin—meeting exact sales requirements

In solving for the program of operation which would yield highest profit, it was assumed that all production could be sold and that any products could be passed up altogether if they did not contribute to the most profit. This solution is valuable because it represents the maximum earning capacity of the production facilities.

In a practical situation, however, it is unlikely that management will find such a program commercially feasible. Usually, there are limits to the flexibility of the demand for products.

Table 5-6 shows how sales restrictions can be introduced to determine the most profitable way of accommodating the actual demand and the amount of potential profit that must be forgone.

Setting up the problem

This variation requires that 1,000 pounds of 3/8-inch-diameter rod must be produced to satisfy outstanding commitments. To introduce this condition, it is necessary to add an equation which says, in effect, "The amount of P_2 produced must be 1,000 pounds." The equation shown on the fourth line of Tableau 1 in Table 5-6 indicates how the condition or restriction of producing exactly 1,000 pounds of Product 2 is put into the problem. This equation is in effect a sales restriction and must be set up in such a way that the method will provide for the production of exactly 1,000 pounds of P_2—no more, no less. To ensure that this condition is met, certain computational steps must be taken. First, the condition must be expressed. This is done simply by the statement $P_2 = 1{,}000$. Adding just this to Matrix 1, however, results in an incomplete diagonal of ones in the basis (the simplex method requires a square basis with a complete diagonal of positive ones). So it becomes necessary to add an artificial variable to complete the basis and permit the computations to proceed. Column P_7 represents the artificial variable in the matrix except that it is not permitted to appear in the final solution as the disposal (idle-time) slack variables can. To prevent its appearance in the final solution, a large negative value or penalty is assigned to prevent it. This value is termed minus M ($-M$), a letter that represents a large unspecified value since a definite penalty is not known. $-M$ is a computational device that behaves and is treated as if it were a number. It frequently helps to think of $-M$ as meaning "*M*ust not happen." In

Table 5-6. Highest Manufacturing Margin Meeting Exact Sales Requirements Drawrod Company

Tableau 1

Margin per unit (in dollars)	Program	P_0	0 P_4	0 P_5	0 P_6	$-M$ P_7	2.10 P_1	4.80 P_2	6.00 P_3		Ratios
0	P_4	700	1	0	0	0	10	23	26	Iteration 1	700/23 = 30.434
0	P_5	550	0	1	0	0	5	13	22		550/13 = 42.307
0	P_6	450	0	0	1	0	7	9	11		450/9 = 50.000
← $-M$	P_7	10	0	0	0	1	0	1	0		10/1 = 10.000 ✓
		$-10M$	0	0	0	0	-2.10	$-M-4.80$	-6.00		

Tableau 2

Margin per unit (in dollars)	Program	P_0	0 P_4	0 P_5	0 P_6	$-M$ P_7	2.10 P_1	4.80 P_2	6.00 P_3		Ratios
← 0	P_4	470	1	0	0	-23	10	0	26	Iteration 2	470/26 = 18 077 ✓
0	P_5	420	0	1	0	-13	5	0	22		420/22 = 19.091
0	P_6	360	0	0	1	-9	7	0	11		360/11 = 32.727
→ 4.80	P_2	10	0	0	0	1	0	1	0		
		48.00	0	0	0	$M+4.80$	-2.10	0	-6.00		

Tableau 3

Margin per unit (in dollars)	Program	P_0	0 P_4	0 P_5	0 P_6	$-M$ P_7	2.10 P_1	4.80 P_2	6.00 P_3	
→ 6.00	P_3	18.077	.038	0	0	$-.885$	.385	0	1	Iteration 3
0	P_5	22.306	$-.846$	1	0	6.461	-3.461	0	0	
0	P_6	161.153	$-.423$	0	1	.731	2.769	0	0	
4.80	P_2	10	0	0	0	1	0	1	0	
Highest profit (in dollars)		156.46	.231	0	0	$M-.208$	.208	0	0	

Highest Manufacturing Margin and Program—Exact Sales (In 100-pound units)

0 units, or	0 lb	Product 1 (P_1 = .5000 OD) at \$2.10 =	0
10.00 units, or	1,000 lb	Product 2 (P_2 = .3750 OD) at 4.80 =	\$ 48.00
18.07 units, or	1,807 lb	Product 3 (P_3 = .2500 OD) at 6.00 =	108.46
Total	2,807 lb		\$156.46

Departmental Load Hours

	P_1	P_2	P_3	Load hours	Available hours	Unused hours
Anneal	0	230	470	700	700	0
Draw	0	130	397.7	527.7	550	22.3
Cut	0	90	198.8	288.8	450	161.2

this particular problem, the slack variable must not appear in the solution if exactly 1,000 pounds of P_2 are to be obtained. To complete the matrix, the value 1,000 is entered in the P_0 Column and a one (1) inserted in the P_2 Column to designate that it is Product 2 requirement that is to be met exactly. It may be helpful in understanding why the 1,000 can be entered in the P_0 Column if it is remembered that the other entries in Matrix 1, or Tableau 1, are also pounds of idle products that have a rate of production of one (1).

Evaluating the answer

The setup of Matrix 1, or Tableau 1, together with the two additional matrices required to obtain a solution, is shown in Table 5-6.

Useful information is obtained by comparing the programs, products, and profits of Table 5-6 with those of Table 5-3. The comparison shows the following variations:

1. Product 1, which appeared in the highest-margin program, does not appear in the highest-margin program to satisfy a commitment of 1,000 pounds of Product 2. Further, the amount of Product 3 produced is less.
2. There are unused hours in the drawing as well as in the cutting operation when sales of Product 2 are met exactly. The total unused hours have increased from 120 to 183.5 hours.
3. Gross margins have decreased from $159.00 to $156.46, a difference of $2.54.

2. Highest manufacturing margin—meeting minimum sales requirements

Frequently production programs are set up to provide capacity for at least a minimum number of units of particular products. For example, it may be necessary to produce at least 1,500 pounds of Product 1 in the current production period. A most profitable program which specifies the profits and products while meeting the minimum sales restrictions can also be worked out by using the simplex method.

Table 5-7 shows how this type of sales restriction can be introduced to accommodate the most profitable way of meeting the minimum requirements that it is desired to meet.

Setting up the method

To make sure that the final program contains at least 1,500 pounds of Product 1 requires several precomputational steps.

Reference to the fourth row in Tableau 1 shows how this restriction is built into the matrix so that the method can take account of it. Again, a minus M $(-M)$ is used to assist the method to arrive at the desired answer, but the reasoning that is followed is somewhat different from

the previous reasoning. The value of $-M$ is still a large unspecified value or penalty, however.

One of the requirements of the simplex method is that all inequality signs must be pointed in the same direction when one is converting inequalities to equalities. The capacity equations were expressed as "less than," whereas the sales of 1,500 pounds are expressed as "more than." For example, the statement of annealing capacity as an inequality is as follows:

$$700 \text{ hours} \geqq 10P_1 + 23P_2 + 26P_3$$

The inequality is pointed to the right ($>$), and was converted to an equation by the addition of another variable, $1P_4$. The minimum-sales requirement expressed as an inequality is as follows:

$$P_1 \geqq 1{,}500$$

The inequality is pointed to the right also but has a different meaning because the terms on each side have been interchanged.

In this particular problem it requires the addition of two unknown variables to reverse the inequality and to make it an equivalent equality.

Since P_1 can be greater than 1,500, it can be made to equal 1,500 by subtracting another term or variable from it. When this is done the inequality becomes the equality:

$$P_1 - P_8 = 1{,}500$$

P_8 can be assigned a rate of production of one (1) and a unit profit of zero (0), the same as the other idle-time variables. However, it is still not possible to proceed with the solution because P_8 has a coefficient of minus one (-1) and cannot appear in the basis or diagonal of ones which must be positive. Consequently, an artificial variable must be added in the same way as the last problem in order to complete the basis. The artificial variable—in this case P_7—is assigned a production rate of one (1) and a cap value of minus M ($-M$) to make certain that it does not appear in the final solution. When the two variables have been added, the expression of minimum sales restriction as an equality for use in the problem is as follows:

$$1{,}500 = 1P_7 - 1P_8 + 1P_1$$

Evaluating the answer

The final program together with the three iterations involved is given in Table 5-7. The information that it contains provides an interesting variation from the most profitable as well as some of the other programs. For example, the increase in Product 1 is obtained at a sacrifice of

Table 5-7. Highest Manufacturing Margin Meeting Minimum Sales Requirements
Drawrod Company

Margin per unit (in dollars)	Program	P_0	0 P_4	0 P_5	0 P_6	$-M$ P_7	0 P_8	2.10 P_1	4.80 P_2	6.00 P_3		Ratios
Tableau 1												
0	P_4	700	1	0	0	0	0	10	23	26	Iteration 1	700/10 = 70.000
0	P_5	550	0	1	0	0	0	5	13	22		550/5 = 110.00
0	P_6	450	0	0	1	0	0	7	9	11		450/7 = 64.685
← $-M$	P_7	15	0	0	0	1	−1	1	0	0		15/1 = 15.000 ✓
		$-15M$	0	0	0	0	M	$-M - 2.10$	−4.80	−6.00		
Tableau 2												
← 0	P_4	550	1	0	0	−10	10	0	23	26	Iteration 2	550/26 = 21.154 ✓
0	P_5	475	0	1	0	−5	5	0	13	22		475/22 = 21.591
0	P_6	345	0	0	1	−7	7	0	9	11		345/11 = 31.364
→ 2.10	P_1	15	0	0	0	1	−1	1	0	0		
		31.50	0	0	0	$M + 2.10$	−2.10	0	−4 80	−6.00		

Tableau 3

→ 6 00	P_3	21.154	.038	0	0	−.385	.384	0	.885	1
0	P_5	9.617	−.846	1	0	3.461	−3 461	0	−6.461	0
0	P_6	112.308	−.423	0	1	−2 769	2 679	0	−.731	0
2.10	P_1	15.000	0	0	0	1	−1	1	0	0
Highest profit (in dollars)		158 42	.231	0	0	M − 208	.208	0	.508	0

Iteration 3

Highest Manufacturing Margin and Program—Minimum Sales Requirements
(In 100-pound units)

15 0 units, or	1,500 lb	Product 1 (P_1 = .5000 OD) at $2.10 =	$ 31.50
0 units, or	0 lb	Product 2 (P_2 = .3750 OD) at 4 80 =	0
21 15 units, or	2,115 lb	Product 3 (P_3 = .2500 OD) at 6.00 =	126.92
Total	3,615 lb		$158 42

Departmental Load Hours

	P_1	P_2	P_3	Load hours	Available hours	Unused hours
Anneal	150	0	550	700	700	0
Draw	75	0	465.4	540.4	550	9.6
Cut	105	0	232.7	337.7	450	112.3

Product 3. Product 2 is not contained in either program. The decrease in profit amounts to $.58, and there is a small increase of 1.9 hours in the number of unused hours.

COMPARING ALTERNATIVE PROGRAMS

Table 5-8 shows a comparison of the various programs that have been computed for the consideration of the management of the Drawrod Company.

In every solution—except for highest material turnover—the limiting

Table 5-8. Comparison of Various Programs
Drawrod Company

Product	Highest manufacturing margin	Least idle time	Highest material turnover	Meeting exact sales requirements	Meeting minimum sales
1	12 20	12 20	64 28	0	15.00
2	0	0	0	10 00	0
3	22.23	22.23	0	18.07	21.15
Total production	34.43	34.43	64 28	28.07	36.15
Gross manufacturing profit (in dollars)	159.00	159.00	135.00	156 46	158.42
Anneal Draw Cut	700 * 550 * 330	700 * 550 * 330	642 8 321 4 450.0 *	700 * 527.7 288.8	700 * 540.4 337.7
Total hours required	1,580	1,580	1,414.2	1,516.5	1,578.1
Unused hours	120	120	286	184	122
Total available hours	1,700	1,700	1,700	1,700	1,700

Production bottleneck.

or bottleneck department is Anneal. Obviously management should plan to increase annealing capacity. If it is assumed that the programs of meeting exact and minimum sales are the programs most likely to be followed in practice, then profits will be increased \$.231 for each hour that is added to annealing capacity. The value is read directly from the P_4 Column and last row of Tableau 3 in each program. The value of \$.231 per unit will hold up to the time that unused capacity in Draw and Cut is utilized or until perfect balance is reached among all departments. It is, therefore, possible to determine directly from the calculations the value of increasing limiting or bottleneck capacity for the existing product mix.

The amount of *each* product that will provide the best use of capacities according to the selected objective is given in each program and is specific. This information can be converted in turn to departmental loads so that planning for operations and man power can be realistic in terms of requirements.

The number of unused hours suggests that methods improvements that increase rates of production and new products that use the same facilities are worth investigation.

Depending upon the program selected to be followed, the specific profit, products, and program provide management with a goal or target to which actual performance can be compared as a basis for control.

SOLVING FOR VARIOUS OTHER OBJECTIVES

In the first chapter, it was pointed out that linear-programming methods provide a way to calculate optimum, or best, answers to certain kinds of problems.

Optimum, or best, answers can be either maximum or minimum. For example, maximum, or most profit, is an optimum objective, and so is minimum, or least time. From the standpoint of the simplex method there is no basic difference between maximizing and minimizing an objective or functional. There are a number of relationships between the two that are worth noting, however.

For example, when we are minimizing it is possible to restate the objective or functional and solve it as a *maximizing* problem or to leave the objective unchanged and solve it as a *minimizing* problem. There are cases when it has been advantageous to do one or the other.

The procedure for converting to a maximizing problem by restating the objective is straightforward. Usually a cost-minimization problem can be changed to a profit-maximization problem as follows:

1. Take the highest cost for each situation as a base. Regard this as a *no*-profit program.

2. Set up each alternate plan as a profit program. The amount of the profit for that program will be the difference obtained by comparing the alternate to the base. This sets up the problem as a profit-maximization problem.

The simplex method also makes it possible to treat a minimum objective as if it were a negative maximum objective. Therefore, an objective equation can be expressed as a minimum equation by multiplying it by a minus one (-1).

Irrespective of whether the functional or objective equation is to be maximized or minimized, the computational routine is the same once the problem has been set up. The steps in setting up the problem are also the same. It is necessary to construct a first basic feasible solution from the equations and inequations that describe the problem conditions and restrictions. The first basic solution is then replaced by another basic feasible solution which improves the objective or functional equation. The improvement process is continued, iteration by iteration, by calculating the incremental change in the objective equation caused by the addition to the basis of an incremental amount of one of the unknowns. An optimum, or best, answer is reached when the next increment is either zero or negative for all possible incremental additions.

In maximizing an objective equation, the calculations are carried forward until all values in the base row are zero or positive. In minimizing, the same condition is true provided that the functional, or objective, equation was multiplied by a minus one (-1).

There is another way of obtaining minimum answers by using the same computational procedure. It involves changing the way in which values are selected at each iteration for improving the functional, or objective, equation. The process is to select and bring into solution the largest positive value at each iteration and proceed until all values in the base row are zero or negative. This process of selection is exactly the reverse of that followed in maximizing.

When this process is to be followed, it is not necessary to multiply the objective equation by a minus one (-1) in order to minimize. The process itself will minimize. The principal advantage of this procedure is that it is somewhat simpler because there are fewer negative signs in the calculations.

SOLVING FOR LEAST TRANSPORTATION COSTS: THE DISHOMATIC COMPANY

A number of applications have been carried out successfully in which the objective was to maximize. In general, there seem to be a growing number of these cases. This has been especially true in the "cracking-type" problem in which one resource has been processed or "cracked"

into a number of end products. Crude oil in a refinery, milk in a creamery, coal tar in a distilling plant, soy beans in a processing plant, and loans at a bank are typical examples.

There have been a number of minimizing applications also—particularly in the areas of minimizing transportation costs, selecting livestock feeds that satisfy minimum nutritional requirements, and building up the soil on a farm to meet specified fertility requirements.

The steps involved in minimizing an objective can be best demonstrated by stating and solving a problem. The particular problem selected for demonstration involves minimizing transportation costs for the Dishomatic Company.

The Dishomatic Company

The Dishomatic Company manufactures automatic dishwashers in three widely separated factories located on the Eastern seaboard. The company owns and operates six warehouses throughout Atlantic coastal states in order to supply the market for their product. Each warehouse can be supplied by one of the three plants using company trucks. The company manufactures for stock in the warehouses. At the present time, the manufacturing capacity of the factories exceeds the requirements of the warehouses.

PROBLEM: To assign the production of the factories to the warehouses at least transportation costs so as to meet the stock requirements of each warehouse for the time being considered.

GIVEN: *The warehouse requirements for the period*

Warehouse I: 100 dishwashers
Warehouse II: 400 dishwashers
Warehouse III: 600 dishwashers
Warehouse IV: 200 dishwashers
Warehouse V: 200 dishwashers

Total requirement for the period is 1,500 dishwashers.

The factory capacity for the period

Factory A: 1,000 dishwashers
Factory B: 300 dishwashers
Factory C: 700 dishwashers

Total factory capacity for the period is 2,000 dishwashers.

Setting up the problem

The following points should be considered in setting up the problem for solution by the simplex method:

1. Each warehouse can be supplied from each factory. This means that there are $5 \times 3 = 15$ activities or possibilities (P).

Table 5-9. Schedule of Freight Costs in Dollars per Dishwasher Using Company Trucks

Factory	Warehouse				
	I	II	III	IV	V
A	30	32	25	50	23
B	40	10	12	21	25
C	21	20	50	18	15

Table 5-10. Number of Activities

Factory	Warehouse				
	I	II	III	IV	V
A	P_1	P_2	P_3	P_4	P_5
B	P_6	P_7	P_8	P_9	P_{10}
C	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}

The possibility of supplying Warehouse I from Factory A is P_1.

2 The objective or functional equation is as follows·

$$Z = 30P_1 + 32P_2 + 25P_3 + 50P_4 + 23P_5 + 40P_6 + 10P_7 + 12P_8 + 21P_9 + 25P_{10} + 21P_{11} + 20P_{12} + 50P_{13} + 18P_{14} + 15P_{15}$$

where the constant values are the costs per unit for each activity or possibility (P) and the Ps are the unknowns for which values are sought that minimize the functional or objective.

3. Since the requirements at each warehouse are to be met exactly, (Warehouse I = 10, Warehouse II = 40, and the like), the problem must be set up so that the method will provide for that condition in the final answer. This is done by placing a large value or penalty designated as $-M$ over the slack variables of the warehouse requirements equations. The $-M$ over the slack variable means that the penalty for having slack is so great that there will not be any and therefore that the requirements will be met exactly.

4. The values in the Cap over the slack variables of the resource or factory equations are zero. The zero indicates that the most desirable or least costly condition occurs when nothing or zero quantity is shipped.

Since factory capacity is 500 units greater than warehouse capacity, there will be 500 units of slack or unused factory capacity.

The first and successive tableaus are given in Table 5-11.

In this particular solution, the functional equation was multiplied by a minus one (-1) and the computations carried forward by selecting the largest negative value at each iteration. The process was continued until the Base Row showed all values to be positive or zero. $M - 24$ is a positive value, even though -24 is contained in it. By definition M is a very large number. Subtracting 24 from it will leave it positive.

This problem could have been solved and the same answer obtained by expressing the functional positively (without the minus sign) and selecting the largest positive value to come in at each iteration until all values in the Base Row became negative or zero.

The least-cost program totals $28,900 in freight costs. Allocating the production of the factories according to the least-cost program shown in Table 5-11 will result in the minimum-cost value.

***Table* 5-11. Least Transportation Costs**
The Dishomatic Company

Cost per unit (in dollars)	Program	P_0	0 P_{16}	0 P_{17}	0 P_{18}	$-M$ P_{19}	$-M$ P_{20}	$-M$ P_{21}	$-M$ P_{22}	$-M$ P_{23}	-30 P_1	-32 P_2	-25 P_3
0	P_{16}	1,000	1								1	1	1
← 0	P_{17}	300		1									
0	P_{18}	700			1								
$-M$	P_{19}	100				1					1		
$-M$	P_{20}	400					1					1	
$-M$	P_{21}	600						1					1
$-M$	P_{22}	200							1				
$-M$	P_{23}	200								1			
		$-1{,}500M$	0	0	0	0	0	0	0	0	$-M+30$	$-M+32$	$-M+25$
0	P_{16}	1,000	1								1	1	1
→ -10	P_7	300	0	1	0	0	0	0	0	0	0	0	0
0	P_{18}	700			1								
$-M$	P_{19}	100				1					1		
$-M$	P_{20}	100		-1			1					1	
$-M$	P_{21}	600						1					1
$-M$	P_{22}	200							1				
← $-M$	P_{23}	200								1			
		$-1{,}200M$ $-3{,}000$	0	$M-10$	0	0	0	0	0	0	$-M+30$	$-M+32$	$-M+25$
0	P_{16}	1,000	1								1	1	1
-10	P_7	300		1									
0	P_{18}	500			1					-1			
$-M$	P_{19}	100				1					1		
$-M$	P_{20}	100		-1			1					1	
$-M$	P_{21}	600						1					1
← $-M$	P_{22}	200							1				
→ -15	P_{15}	200	0	0	0	0	0	0	0	1	0	0	0
		$-1{,}000M$ $-6{,}000$	0	$M-10$	0	0	0	0	0	$M-15$	$-M+30$	$-M+32$	$-M+25$

Table 5-11. Least Transportation Costs (*continued*)
The Dishomatic Company

−50 P_4	−23 P_5	−40 P_6	−10 P_7	−12 P_8	−21 P_9	−25 P_{10}	−21 P_{11}	−20 P_{12}	−50 P_{13}	−18 P_{14}	−15 P_{15}
1	1										
		1	1	1	1	1					
							1	1	1	1	1
		1					1				
			1					1			
				1					1		
1					1					1	
	1					1					1
$-M+50$	$-M+23$	$-M+40$	$-M+10$	$-M+12$	$-M+21$	$-M+25$	$-M+21$	$-M+20$	$-M+50$	$-M+18$	$-M+15$
1	1		0								
0	0	1	1	1	1	1	0	0	0	0	0
			0				1	1	1	1	1
		1	0				1				
		−1	0	−1	−1	−1		1			
			0	1					1		
1			0		1					1	
	1		0			1					1
$-M+50$	$-M+23$	30	0	2	11	15	$-M+21$	$-M+20$	$-M+50$	$-M+18$	$-M+15$
1	1										0
		1	1	1	1	1					0
	−1					−1	1	1	1	1	0
		1					1				0
		−1		−1	−1	−1		1			0
				1					1		0
1					1					1	0
0	1	0	0	0	0	1	0	0	0	0	1
$-M+50$	8	30	0	2	11	M	$-M+21$	$-M+20$	$-M+50$	$-M+18$	0

Table 5-11. Least Transportation Costs (*continued*)
The Dishomatic Company

Cost per unit (in dollars)	Program	P_0	0 P_{16}	0 P_{17}	0 P_{18}	$-M$ P_{19}	$-M$ P_{20}	$-M$ P_{21}	$-M$ P_{22}	$-M$ P_{23}	-30 P_1	-32 P_2	-25 P_3
0	P_{16}	1,000	1								1	1	1
−10	P_7	300		1									
0	P_{18}	300			1				−1	−1			
$-M$	P_{19}	100				1					1		
← $-M$	P_{20}	100		−1			1					1	
$-M$	P_{21}	600						1					1
→ −18	P_{14}	200	0	0	0	0	0	0	1	0	0	0	0
−15	P_{15}	200								1			
		$-800M$ $-9{,}600$	0	$M-10$	0	0	0	0	$M-18$	$M-15$	$-M+30$	$-M+32$	$-M+25$
0	P_{16}	1,000	1								1	1	1
−10	P_7	300		1									
0	P_{18}	200		1	1		−1		−1	−1		−1	
← $-M$	P_{19}	100				1					1		
→ −20	P_{12}	100	0	−1	0	0	1	0	0	0	0	1	0
$-M$	P_{21}	600						1					1
−18	P_{14}	200							1				
−15	P_{15}	200								1			
		−700 −11,600	0	10	0	0	$M-20$	0	$M-18$	$M-15$	$-M+30$	12	$-M+25$
0	P_{16}	1,000	1								1	1	1
−10	P_7	300		1									
← 0	P_{18}	100		1	1	−1	−1		−1	−1	−1	−1	
→ −21	P_{11}	100	0	0	0	1	0	0	0	0	1	0	0
−20	P_{12}	100		−1			1					1	
$-M$	P_{21}	600						1					1
−18	P_{14}	200							1				
−15	P_{15}	200								1			
		$-600M$ $-13{,}700$	0	10	0	$M-21$	$M-20$	0	$M-18$	$M-15$	9	12	$-M+25$

Table 5-11. **Least Transportation Costs** (*continued*)
The Dishomatic Company

−50 P_4	−23 P_5	−40 P_6	−10 P_7	−12 P_8	−21 P_9	−25 P_{10}	−21 P_{11}	−20 P_{12}	−50 P_{13}	−18 P_{14}	−15 P_{15}
1	1									0	
		1	1	1	1	1				0	
−1	−1				−1	−1	1	1	1	0	
		1					1			0	
		−1		−1	−1	−1		1		0	
				1					1	0	
1	0	0	0	0	1	0	0	0	0	1	0
	1					1				0	1
32	8	30	0	2	$M-7$	M	$-M+21$	$-M+20$	$-M+50$	0	0
1	1							0			
		1	1	1	1	1		0			
−1	−1	1		1			1	0	1		
		1					1	0			
0	0	−1	0	−1	−1	−1	0	1	0	0	0
				1				0	1		
1					1			0		1	
	1					1		0			1
32	8	$-M+50$	0	$-M+22$	13	20	$-M+21$	0	$-M+50$	0	0
1	1						0				
		1	1	1	1	1	0				
−1	−1			1			0		1		
0	0	1	0	0	0	0	1	0	0	0	0
		−1		−1	−1	−1	0	1			
				1			0		1		
1					1		0			1	
	1					1	0				1
32	8	29	0	$-M+22$	13	20	0	0	$-M+50$	0	0

Table 5-11. Least Transportation Costs (*continued*)
The Dishomatic Company

Cost per unit (in dollars)	Program	P_0	0 P_{16}	0 P_{17}	0 P_{18}	$-M$ P_{19}	$-M$ P_{20}	$-M$ P_{21}	$-M$ P_{22}	$-M$ P_{23}	-30 P_1	-32 P_2	-25 P_3
0	P_{16}	1,000	1								1	1	1
−10	P_7	200			−1	1	1		1	1	1	1	
→ −12	P_8	100	0	1	1	−1	−1	0	−1	−1	−1	−1	0
−21	P_{11}	100				1					1		
−20	P_{12}	200			1	−1			−1	−1	−1		
← $-M$	P_{21}	500		−1	−1	1	1	1	1	1	1	1	1
−18	P_{14}	200							1				
−15	P_{15}	200								1			
		$-500M$ −15,900	0	$M-12$	$M-22$	1	2	0	4	7	$-M+31$	$-M+34$	$-M+25$
0	P_{16}	500	1	1	1	−1	−1	−1	−1	−1			0
−10	P_7	200			−1	1	1		1	1	1	1	0
−12	P_8	100		1	1	−1	−1		−1	−1	−1	−1	0
−21	P_{11}	100				1					1		0
−20	P_{12}	200			1	−1			−1	−1	−1		0
→ −25	P_3	500	0	−1	−1	1	1	1	1	1	1	1	1
−18	P_{14}	200							1				0
−15	P_{15}	200								1			0
Least cost (in dollars)		28,400	0	13	3	$M-24$	$M-23$	$M-25$	$M-21$	$M-18$	6	9	0

Least-cost Program

Factory	Warehouse I	II	III	IV	V	Unused	Total
A			500			500	1,000
B		200	100				300
C	100	200		200	200		700
Total	100	400	600	200	200	500	2,000

Table 5-11. Least Transportation Costs (*continued*)
The Dishomatic Company

-50 P_4	-23 P_5	-40 P_6	-10 P_7	-12 P_8	-21 P_9	-25 P_{10}	-21 P_{11}	-20 P_{12}	-50 P_{13}	-18 P_{14}	-15 P_{15}
1	1			0							
1	1	1	1	0	1	1			-1		
-1	-1	0	0	1	0	0	0	0	1	0	0
		1		0			1				
-1	-1	-1		0	-1	-1		1	1		
1	1			0							
1				0	1					1	
	1			0		1					1
$-M+54$	$-M+30$	29	0	0	13	20	0	0	28	0	0
1	1	1	1		1	1			-1		
-1	-1			1					1		
		1					1				
-1	-1	-1			-1	-1		1	1		
1	1	0	0	0	0	0	0	0	0	0	0
1					1					1	
	1					1					1
29	5	29	0	0	13	20	0	0	28	0	0

CHAPTER 6

The Ratio-analysis Method

BACKGROUND AND INTRODUCTION TO THE RATIO-ANALYSIS METHOD

Ratio Analysis is a new method and approach to the solution of linear-programming-type problems.[1] Because of its newness it has had limited application, and as a method it has not been explored fully. On the basis of the evidence to date, however, it has shown good potential for simplifying difficult problems. It has proved potentially valuable in acquiring a better grasp of the simplex method.

The modi and index methods were developed to meet specific needs—to solve certain problems or reduce calculation time for others—and their uses have since been widely extended. The ratio-analysis method, on the other hand, is the development of an idea, or an approach to programming generally. It changes the form of the problem information in such a way as to make possible direct logical comparisons of alternatives. This greatly simplifies calculation. The method cannot be used for all types of problem. But—for the appropriate type—it will rapidly give an optimum answer or greatly reduce the number of steps in the simplex method. Ratio Analysis is defined as follows:

> Ratio Analysis is a procedure for allocating limited resources most effectively among competing demands where there are differences in unit profit (cost or other benefit) and in capacity required to satisfy a unit of demand.

It seems that almost everyone who makes decisions is confronted with the type of problem referred to in the definition. For example, the foreman or scheduling clerk has to decide which jobs will go on the ideal machines and which ones will go on the non-ideal machines. The chief engineer has to select or recommend the type of work to be done—the development of a new product, the improvement of the design of an old

product, or some other problem. The president of the company may have to decide between building a new plant, developing a new product, or intensifying the advertising program.

In many business situations the demands for resources exceed the amount of available resources. So it frequently becomes necessary to select one of several desirable courses of action. The cost of doing this is the loss of opportunity of doing other things. Economists call this cost an opportunity cost.

Some people learn to make such selections by intuitive processes. In some cases making the right selection is mainly a problem of organizing and interpreting facts. Sometimes the problem is one of weighing and balancing intangibles and coming up with an integrated program—in these cases there is no substitute for judgment, knowledge, and experience. It is desirable, however, to base as many of the decisions and as much of each decision as possible on an analysis of facts. The expansion of scientific management has been based on using more facts and less integration of experience and knowledge.

Much time has been spent developing ways of collecting, organizing, and interpreting facts for making decisions. Linear programming has proved to be extremely useful for solving certain kinds of industrial problems, for it provides a precise way of using statements of limitations, such as "not more than" and "not less than," in mathematical computations. Ratio analysis provides a way of solving some problems of this kind.

A RATIO-ANALYSIS PROBLEM: THE MATHAND COMPANY

This tool or technique can be an aid in a wide area of decision making. Its usefulness covers a broad range of problems. A specific problem in which a number of different demands compete for limited amounts of resources will demonstrate this. In this problem the demands can be for different products which earn varying amounts of profit and require varying amounts of the available capacity of the several departments in the manufacturing division of the business. The objective is to meet as many of these demands as possible in the way most beneficial over all.

The Mathand Company

A company makes material-handling equipment—traveling cranes, jib cranes, fork lift trucks, pallet elevators, and stacking conveyors. The sales have exceeded the capacity of the plant, and management feels that the company cannot finance an expansion program at this time. The sales manager proposes that they concentrate their activities on the products that will bring in the most profits from their present facilities. This

brings up the question of which products the company should make. The manufacturing division has six main departments: Foundry, Forge Shop, Weld Shop, Machine Shop, Finishing, and Erection Shop. Each of the products requires a certain amount of the capacity of each department.

The problem information is listed in Table 6-1. It consists of the capacity of each department in man-hours per month, the number of man-hours required by an average unit of each product in each department, and the average profit per unit of product.

Table 6-1. Problem Information

Products and Average Profit per Unit; Departments and Capacity in Man-hours per Month: Man-hours Required in Each Department to Make One Average Unit of Each Product

Products		Traveling cranes	Jib cranes	Fork lift trucks	Pallet elevators	Stacking conveyors
Profit per unit (in dollars)		6,000	300	750	1,200	300
Departments	Capacity (in man-hours per month)	Man-hours required per unit of product				
Foundry	2,000	800	20	20	120	30
Forge Shop	1,000	200	10	15	30	20
Weld Shop	1,000	300	20	40	45	10
Machine Shop	8,000	2,400	40	240	320	160
Finishing	2,000	400	30	50	80	40
Erection Shop	6,000	900	60	240	180	120

STEPS FOR SETTING UP AND SOLVING A PROBLEM

The use of the ratio-analysis method involves a number of steps which follow a definite pattern. The steps in the order in which they should be taken are given below.

1. Form a ratio-analysis matrix

The first step is to arrange the problem data in a table or matrix to facilitate solution. As in the other methods, a matrix consists of the co-

efficients of a family of related equations and inequations. With the information in Table 6-1, the steps for doing this are as follows:

1. Find a common unit of measure for the resources and demands. The first and frequently a most important step in the use of ratio analysis is to find a common unit of measure for relating the demands and the resources to be used in meeting these demands.

In this case the common unit of measure is man-hours. The available man-hours per month in each department are given. This is a measure of the available resources. The benefits are the profits that can be made by supplying the demand for the several products. Table 6-1 gives the number of hours it takes to make each of the products in each of the departments. In many cases, the problem is well on the way to being solved when it can be set up with a common unit of measure, as in Table 6-1.

2. Set up the unknowns

The unknowns are the amounts of each product to be made from the available amount of productive capacity to maximize profits. They are designated by the following letters.

v = the number of traveling cranes that should be made per month
w = the number of jib cranes to be made per month
x = the number of fork lift trucks to be made per month
y = the number of pallet elevators to be made per month
z = the number of stacking conveyors to be made per month

In ratio analysis the unknowns represent the answers that are wanted. They will have either a positive or a zero value.

3. Organize the information into a table of inequations

The facts given in Table 6-1, when properly related, can be expressed in the form of inequations for use in the matrix.

The first fact is the amount of capacity in man-hours required in each department to make an average unit of each product. For example, it takes 800 of the Foundry man-hours to make one traveling crane.

The second fact given in the table is the amount of capacity in man-hours that is available in each department. For example, the Foundry has 2,000 man-hours available for the month.

The third fact, related to the first two, is that the amount of products that can be made is limited by the capacity of one or more of the departments. If the one department were the Foundry, for example, the total number of each product made per month multiplied by the number of hours required to make one unit of the product cannot exceed the available capacity in foundry man-hours. It is not known in advance,

however, whether all of the Foundry capacity will be used in the most profitable program or not. The same statement is true of the other departments, so that the use of capacity must be determined by the method. To do this requires that the possible uses of the capacity of each department are expressed as an inequation containing the products that can be run. For the Foundry, this expression is as follows:

$$800v + 20w + 20x + 120y + 30z \leqq 2{,}000$$

where the letters v, w, x, y, and z represent the quantity of each of the products and the numbers 800, 20, 20, 120, and 30 are the man-hours per product, respectively.

When the capacity of each department is similarly expressed as inequations, they are then arrayed in the matrix form shown in Table 6-2.

Table 6-2. Departmental Capacity Inequations

$$\begin{array}{rcrcrcrcrcl}
800v & + & 20w & + & 20x & + & 120y & + & 30z & \leq & 2{,}000 \\
200v & + & 10w & + & 15x & + & 30y & + & 20z & \leq & 1{,}000 \\
300v & + & 20w & + & 40x & + & 45y & + & 10z & \leq & 1{,}000 \\
2{,}400v & + & 40w & + & 240x & + & 320y & + & 160z & \leqq & 8{,}000 \\
400v & + & 30w & + & 50x & + & 80y & + & 40z & \leq & 2{,}000 \\
900v & + & 60w & + & 240x & + & 180y & + & 120z & \leqq & 6{,}000
\end{array}$$

Each column in Table 6-2 designates a product and the coefficients relating to it. It is important to the success of the calculations that the coefficient of each unknown remains in the same position in the table even when its value changes as the calculations are carried forward.

The final step in setting up the matrix is to place the profit figure for

Table 6-3. Ratio-analysis Matrix

6,000	300	750	1,200	300	
800	20	20	120	30	2,000
200	10	15	30	20	1,000
300	20	40	45	10	1,000
2,400	40	240	320	160	8,000
400	30	50	80	40	2,000
900	60	240	180	120	6,000

each product at the top of its product column in the matrix and omit the product letter notation.

When this step has been completed, the matrix will appear as columns of numbers only as shown in Table 6-3. This array of numbers is the Ratio-analysis matrix. It is the same as *part* of the matrix for the simplex method of linear programming.

4. Unitize the rows of the ratio-analysis matrix

The purpose of this step is to set up each resource as unity—that is, one Machine Shop, one Erection Shop, and the like. This is done by dividing the numbers in *each* row by *their* constant value at the end of the row. When this process is completed, the time to make one unit of a product is expressed as a fraction of the capacity for 1 month. This calculation does not affect the profit per unit for each product. The calculations, when carried out for the problem, appear in Table 6-4.

Table 6-4. Unitized Matrix

6,000	300	750	1,200	300	
.400	.010	.010	.060	.015	1.000
.200	.010	.015	.030	.020	1.000
300	020	.040	.045	.010	1.000
.300	.005	.030	.040	.020	1.000
.200	015	.025	.040	020	1.000
.150	.010	.040	030	.020	1.000

These computations in no way change the validity of the inequations in the First Matrix. Both sides of each inequation have been divided by the same number. For example, in the First Matrix it takes 800 hours of 2,000 available hours per month in the Foundry to make one traveling crane per month. In the Unitized Matrix it takes .4 of the Foundry capacity per month to make one traveling crane per month. There is no difference in the two statements. They state the same fact in different ways.

The Unitized Matrix provides a way of comparing the requirements for manufacturing each product in terms of the capacity of each department. The Unitized Matrix states that it takes .030 of the Machine Shop capacity to make a fork lift truck. It also provides similar information about each product in each department. Since the objective is to earn as much profit as possible, the profit possibilities of each product must

be set up and compared. The process of setting up and converting the information to a comparable basis is termed *equalizing.*

5. Equalize the columns of the ratio-analysis matrix

The comparisons of profit are accomplished by establishing *groups* of each product—selected so that the profit on each group is the same as that for a group of any other product. For example, the profit on one traveling crane is $6,000, and on one jib crane it is $300. Twenty jib cranes will also earn $6,000 profit. On the basis of profit-earning capacity, it will take 20 times as much plant capacity to make twenty jib cranes as it does to make one traveling crane. Therefore, we multiply each number in the jib-crane column by the ratio of profits: 6,000/300, or 20. This changes the numbers in this column as follows:

$$
\begin{aligned}
300 \times 20 &= 6{,}000 \\
.010 \times 20 &= .200 \\
.010 \times 20 &= .200 \\
.020 \times 20 &= .400 \\
.005 \times 20 &= .100 \\
.015 \times 20 &= .300 \\
.010 \times 20 &= .200
\end{aligned}
$$

All the numbers in each column are multiplied by a number which increases to $6,000 the profit for that column. The unitized numbers for fork lift trucks would be multiplied by 6,000/750, or 8.

Similarly, the respective multipliers for pallet elevators and stacking conveyors would be 6,000/1,200, or 5, and 6,000/300, or 20. The numerators for these fractions can be any convenient number. The $6,000 value was selected for this case because it was the least common multiple of the profits. $10,000, $300, or $1 could have been used with equal accuracy. The computations are made easier, however, by using a least common multiple, if the number is not too large. The processing of a unitized matrix in this way is called *equalization,* and the matrix is now said to be a *unitized, equalized matrix.* The example matrix is shown in unitized, equalized form in Table 6-5. This array of numbers is basically the same as in the first Ratio-analysis Matrix (Table 6-3). However, now that direct comparison is possible, facts can be found and comparisons made that were somewhat difficult to see in the First Matrix. A study of the requirements along the rows for the departments will demonstrate this. It can readily be seen that an equalized group of each product does not require an equal part of each department's capacity. An equalized unit of jib cranes requires .400 of the Weld Shop's capacity for a month, and .100 of the Machine Shop's capacity for a month. Similarly, an equalized unit of pallet elevators requires .300 of

Table 6-5. Unitized, Equalized Matrix

	Traveling cranes	Jib cranes	Fork lift trucks	Pallet elevators	Stacking conveyors	
	6,000	6,000	6,000	6,000	6,000	
Foundry	.400	.200	.080	.300	.300	1.000
Forge Shop	.200	.200	.120	.150	.400	1.000
Weld Shop	.300	.400	.320	.225	.200	1.000
Machine Shop	.300	.100	.240	.200	.400	1.000
Finishing	.200	.300	.200	.200	.400	1.000
Erection Shop	.150	.200	.320	.150	.400	1.000

the Foundry's capacity per month and only .150 of the Forge Shop's and Erection Shop's capacity per month. The Unitized, Equalized Matrix provides a clearer picture of the profit-earning capacity of each resource when it is used to make the different products. Now it is possible to start solving the problem, which was to determine which products to manufacture in order to be able to make the most profit per month. There is a definite procedure for doing this. First, determine which product or products will earn the greatest profit if only one product is made. Then determine whether it is more profitable to make a combination of products rather than only one product.

6. Find the product which shows maximum profit potential when only one product is manufactured—find the first key number

The next step is to determine the most profitable product when the company is limited to making only one product. For example, the largest number in the column for traveling cranes is .400 in the Foundry. This means that .400 of the Foundry capacity is used for one profit unit of production. It is possible to make only 1.000 ÷ .400, or 2.5 equalized groups of traveling cranes. Making this quantity of traveling cranes requires *all of the Foundry capacity,* even though 1.000/.200 = 5 traveling-crane groups can be made in the Forge Shop. The bottleneck in the Foundry limits production to 2.5 groups. Two steps are necessary to find the one best product to make. First, *select the largest number in each column.* They are .400, .400, .320, .300, and .400, respectively. A dot has been placed over each of these largest numbers in each column in Table 6-5. Second, *select the smallest of these largest numbers*—this is .300 in the column representing pallet elevators and in the row for the Foundry. This number is circled in Table 6-5. This means that more profit units of pallet elevators can be made than any other product and

that for this product the bottleneck is the Foundry capacity. Hereafter, this smallest of the largest numbers is called the First Key Number. This corresponds to the choice of the key column in the simplex method—the most negative number, or product which adds most profit—and to the choice of the key row as the bottleneck. The simplex key number is the intersection of this greatest-profit per unit column and the bottleneck row. There may be other key numbers in the matrix. If so they will be called the Second Key Number and the Third Key Number. The row and the column in which this number is found will be respectively called the First Key Row and the First Key Column. The rows and columns of the second and third key numbers, if any, will be called the Second and Third Key Rows and Key Columns, respectively.

It is possible to make 1.000 ÷ .300, or 3⅓ equalized groups of pallet elevators. This uses all of the Foundry capacity. The profit potential is 3⅓ × \$6,000, or \$20,000 per month if only this one product is made. This is the most profit that can be earned in 1 month, when the company makes only one product. The most of any one product that can be made is 1.000 divided by the largest number in the column representing the product. The Key Number, or .300, was the smallest of the largest numbers in each column. Therefore, 1.000 ÷ .300 will give a larger answer than 1.000 divided by any other larger number. The result of this division determines the number of groups of equalized profit units that can be made for the chosen product. Now the next step is to determine whether it is possible to increase the profit potential of the plant by making a combination of two or more products.

7. Determine whether the potential profit can be increased by manufacturing a combination of two or more products

Quite frequently it is possible to find a combination of two or more products that will increase the profit potential above that resulting from the manufacture of one product. It is easy to see that this can be true. The single product—pallet elevators—that was selected requires a smaller part of the capacity of the other departments than it does of the Foundry. Or, in other words, more pallet elevators could be made if there were more Foundry capacity. In fact, if there were enough Foundry capacity production could be increased until restricted by the department represented by the next largest number in the column, which is the .225 representing the Weld Shop. If this were the bottleneck the capacity would be 1.000 ÷ .225, or 4⁴⁄₉ equalized profit groups of pallet elevators. This would increase the profit potential per month from \$20,000 to \$26,666.67. These figures show that the Ratio-analysis Matrix provides a way of evaluating the relative merits of capacity increases, of changes in products, of methods improvements, and the like. One of the conditions

of the problem, however, was that the productive capacity could not be increased. To increase profits, therefore, it is necessary to find another product that requires less Foundry capacity per equalized product group than the first product, pallet elevators. Then *some* of this product can be made. Since the newly selected product requires less Foundry capacity per equalized group of products, the first bottleneck (the Foundry) is opened up. However, this new product group, making the same profit, will require more capacity of some other department or departments than the group of pallet elevators because some number in the other columns is equal to or larger than the corresponding number in the First Key Column. So Foundry capacity can be assigned to another product until the capacity of some other department is exhausted.

This is done by looking for a number in the First Key Row that is smaller than the First Key Number. If there is such a number or numbers, the amount of the potential profits can be increased by manufacturing two or more products. There are two such numbers in the problem. They are the .080 in the column representing fork lift trucks and the .200 in the column representing jib cranes.

8. Determine the amount of the potential increases in profits

Start empirically with the smaller of the two numbers. In other words, look into the profit potential of manufacturing a combination of pallet elevators and fork lift trucks. The choice of fork lift trucks may be indicated by the fact that the second bottleneck capacity for pallet elevators is the Weld Shop, which can produce more fork lift trucks than jib cranes. It only takes .080 of the Foundry capacity per month to produce an equalized group of fork lift trucks, while it takes .300 of the Foundry capacity per month to produce an equalized group of pallet elevators. A transfer of .08 of a month's Foundry capacity to fork lift trucks gains one equalized group of fork lift trucks and loses .08/.30, or $\frac{4}{15}$ of a group of pallet elevators. Therefore, the net gain is $(1 - \frac{4}{15})$, or $\frac{11}{15}$ equalized profit groups. This equalized profit, or the profit on an equalized group of products, is \$6,000. Then the dollar gain on transferring .08 of the Foundry capacity per month to one equalized group of fork lift trucks is $(\frac{11}{15} \times \$6{,}000)$, or \$4,400.

The next resource or capacity to be exhausted will be the department other than the Foundry in which these products have the largest combined load. The second heaviest load on pallet elevators is in the Weld Shop, where one equalized group of these products takes .225 of a month's capacity. The heaviest load from the fork lift trucks occurs in the Weld Shop and the Erection Shop. One equalized group of fork lift trucks requires .320 of a month's capacity in each of these departments. The heaviest load from pallet elevators outside the Foundry is in the

Weld Shop, and the fork lift trucks require as much of the Weld Shop's capacity as in any other department. Therefore the Weld Shop will be the second bottleneck when these two products are produced.

These two products will use all of the capacity of the Foundry and the Weld Shop. So it is possible to set up two simultaneous equations that will tell how many equalized groups of each product can be made.

x_e = the number of *equalized* groups of fork lift trucks that will be made

y_e = the number of *equalized* groups of pallet elevators that will be made

And the number of the equalized groups of each of these products multiplied by the time for each group of products uses up the capacity of the Foundry and the Weld Shop. Therefore,

$$.080x_e + .300y_e = 1.000 \text{ Foundry}$$
$$.320x_e + .225y_e = 1.000 \text{ Weld Shop}$$

These equations can be solved by determinants, or algebraically. When the equations are solved simultaneously the answers are as follows:

$$\text{Fork lift trucks} \quad x_e = \frac{.225 - .300}{.018 - .096} = \frac{-.075}{.078} = \frac{25}{26}$$

$$\text{Pallet elevators} \quad y_e = \frac{.080 - .320}{.018 - .096} = \frac{-.240}{.078} = \frac{40}{13}$$

Thus the total profit is now $4\frac{1}{26} \times \$6{,}000$, or \$24,231 per month.

The total equalized-profit groups with one product was $3\frac{1}{3}$. The gain from making two products is $\frac{105}{26} - \frac{10}{3}$, or $\frac{315}{78} - \frac{260}{78}$, or $\frac{55}{78}$ of an equalized product group. The additional profits on this are $\frac{55}{78} \times \$6{,}000$, or \$4,231.

9. Determine whether it is possible to further increase the profit potential by including another product in the manufacturing group

It will be possible to increase the profit potential if there is any other product that requires less Weld Shop and Foundry capacity than the products already in the group. For then it will be possible to gain capacity to produce more groups of equalized products. In other words, the Foundry Row is the First Key Row, and the Weld Shop is the Second Key Row. The pallet-elevator column is the First Key Column and the fork lift–truck column is the Second Key Column.

There may be another product that it will pay to include in the group of products to be manufactured. If so, then *there must be a col-*

umn where a number in a key row is smaller than one of the key column numbers in the same key row. Preferably, all key-row numbers in the chosen column should be equal to, or less than, *one* of the key column numbers in the same key row. The Stacking Conveyor Column meets these requirements. Its number in the First Key Row is .300 which is the same as the Key Number. Its number in the Second Key Row is .200 which is smaller than either of the key column numbers .320 and .225 in this row. So it is possible to increase the potential profit-earning capacity of the plant by including the stacking conveyors in the products that are to be manufactured.

10. Determine the third limiting resource

The next step is to determine which of the departments will prove to be the third limiting resource. It can be seen by inspection that the Forge Shop or the Finishing Department is not the third limiting resource. The limiting department will be the one that requires the most time to make these three products. The Forge Shop and the Finishing Shop require less time to make these products than the Machine Shop. Therefore, the Machine Shop will be a limit on production before either of the other two departments.

The stacking conveyors and the fork lift trucks require as much as or more of the Erection Shop monthly capacity than they do of the Machine Shop monthly capacity. However, the pallet elevators require much less of the Erection Shop capacity than they do of the Machine Shop capacity per month. Therefore, either the Machine Shop or the Erection Shop will be the third limiting restriction. (If the .320 figure for fork lift trucks in the Erection Shop had been less than the .240 in the Machine Shop, the Erection Shop would have been eliminated as a bottleneck and the problem solved with the Machine Shop as the third limiting restriction.)

A comparison of the simplex and ratio-analysis methods will be helpful in understanding what is done next. Both the simplex and the ratio-analysis matrices contain a number of essential and unessential facts. Both of the procedures are planned to separate the essential facts from the unessential facts. When this is done the result is a set of simultaneous equations that are in the matrix.[2] In the simplex matrix the coefficients of the unknowns for the equations are found where the key columns intersect the key rows. The negative numbers in the base indicate which items should be included in the equations. The smallest positive quotient picks the restriction for each product. The simplex routine in effect selects a set of simultaneous equations and solves them. The simplex solution to this problem is presented in Table 6-6 for comparison.

[2] This fact was first called to the authors' attention by G. M. W. Sebus, of Amsterdam, while they were on assignment in the Netherlands in 1954.

Table 6-6. Most Profitable Product Mix

Profit per unit (in dollars)	Program product		0	0	0	0	0	0	Traveling cranes 6,000	Jib cranes 300	Fork lift trucks 750	Pallet elevators 1,200	Stacking conveyors 300	
		P_0	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_1	P_2	P_3	P_4	P_5	
← 0	P_6	200	1	0	0	0	0	0	80	2	2	12	3	Iteration 1
0	P_7	100	0	1	0	0	0	0	20	1	1.5	3	2	
0	P_8	100	0	0	1	0	0	0	30	2	4	4.5	1	
0	P_9	800	0	0	0	1	0	0	240	4	24	32	16	
0	P_{10}	200	0	0	0	0	1	0	40	3	5	8	4	
0	P_{11}	600	0	0	0	0	0	1	90	6	24	18	12	
		0	0	0	0	0	0	0	−6,000	−300	−750	−1,200	−300	
→ 6,000	P_1	2.5	1/80	0	0	0	0	0	1	1/40	1/40	3/20	3/80	Iteration 2
0	P_7	50	−1/4	1	0	0	0	0	0	1/2	1	0	5/4	
← 0	P_8	25	−3/8	0	1	0	0	0	0	5/4	13/4	0	−1/8	
0	P_9	200	−3	0	0	1	0	0	0	−2	18	−4	7	
0	P_{10}	100	−1/2	0	0	0	1	0	0	2	4	2	5/2	
0	P_{11}	375	−9/8	0	0	0	0	1	0	15/4	87/4	9/2	69/8	
		15,000	75	0	0	0	0	0	0	−150	−600	−300	−75	

← 6,000	P_1	30/13	1/65	0	−1/130	0	0	0	1	[illegible]	0	3/20	1/26	
0	P_7	550/13	−7/52	1	−4/13	0	0	0	0	3/26	0	0	67/52	
→ 750	P_3	100/13	−3/26	0	4/13	0	0	0	0	5/13	1	0	−1/26	
0	P_9	800/13	−12/13	0	−72/13	1	0	0	0	−116/13	0	−4	100/13	Iteration 3
0	P_{10}	900/13	−1/26	0	−16/13	0	1	0	0	6/13	0	2	69/26	
0	P_{11}	2 700/13	18/13	0	−87/13	0	0	1	0	−60/13	0	9/2	123/13	
		255 000/13	15/13	0	2 400/13	0	0	0	0	1 050/13	0	−300	−1 275/13	
→ 1 200	P_4	200/13	20/195	0	−20/390	0	0	0	20/3	20/195	0	1	20/78	
0	P_7	550/13	−7/52	1	−4/13	0	0	0	0	3/26	0	0	67/52	
750	P_3	100/13	−3/26	0	4/13	0	0	0	0	5/13	1	0	−1/26	
← 0	P_9	1 600/13	−20/39	0	−224/39	1	0	0	0	−332/39	0	0	340/39	Iteration 4
0	P_{10}	500/13	−19/78	0	−44/39	0	1	0	−40/3	[illegible]	0	0	167/78	
0	P_{11}	1 800/13	12/13	0	−84/13	0	0	1	−30	−66/13	0	0	108/13	
		315 000/13	475/13	0	2 200/13	0	0	0	2 000	1 450/13	0	0	−275/13	
1,200	P_4	200/17	2/17	0	2/17	−1/34	0	0	20/3	6/17	0	1	0	
0	P_7	410/17	1/17	1	46/85	201/1 360	0	0	0	467/340	0	0	0	
750	P_3	140/17	−2/17	0	24/85	3/680	0	0	0	59/170	1	0	0	
→ 300	P_5	240/17	−1/17	0	−56/85	39/340	0	0	0	83/85	0	0	1	Iteration 5
0	P_{10}	140/17	2/17	0	24/85	−167/680	1	0	−40/3	399/170	0	0	0	
0	P_{11}	360/17	24/17	0	−84/85	−81/85	0	1	−30	258/85	0	0	0	
Highest profit (in dollars)		\$417 000/17	600/17	0	2 640/17	165/84	0	0	2 000	1,545/17	0	[illegible]	0	

Most profitable program $\left\{\begin{array}{l} P_4\text{—pallet elevators—}11\tfrac{13}{17} \\ P_3\text{—fork lift trucks—}8\tfrac{4}{17} \\ P_5\text{—stacking conveyors—}14\tfrac{2}{17} \end{array}\right\}$ = \$24,529 7/17

Sometimes the ratio-analysis procedure will select these equations, and sometimes it will not. This problem was set up so that it would select these equations and provide the most profitable answer. If the ratio-analysis method will not solve the matrix, at least it will decrease both the size and the number of iterations in the simplex method. Comparing the simplex matrix with ratio-analysis matrix shows a number of differences. First, an identity is not needed when a ratio-analysis matrix can be set up. Second, a brief inspection shows that the Forge Shop and the Finishing Department will never be restrictions. Consequently, they do not need to be considered and can be omitted from the matrix. It is also possible to determine by inspection that neither the traveling cranes nor the jib cranes can be made profitably. So they can be dropped from the matrix also. This decreases the size of the matrix from 7×12 to 5×4, which reduces the number of calculations.

To return to the sample problem, there are two different ways of finding out which of these departments—the Erection Shop or the Machine Shop—is the limiting resource. One way is to *solve two sets of three simultaneous equations* that express the capacity of the three departments. Each set of simultaneous equations will have an equation representing one of the departments being checked and will include the Foundry and Weld Shop. Then the set of equations with smaller total values of the unknowns is the restricting department. The first set of equations using unitized, equalized values to be solved which includes the Machine Shop is as follows:

Fork lift trucks		Pallet elevators		Stacking conveyors	
$.08x_e$	+	$.30y_e$	+	$.30z_e$	= 1.000 Foundry
$.32x_e$	+	$.225y_e$	+	$.20z_e$	= 1.000 Weld Shop
$.24x_e$	+	$.20y_e$	+	$.40z_e$	= 1.000 Machine Shop

x_e means equalized groups of item x.

This set of equations solves to:

For fork lift trucks	$x_e = 35/34$
For pallet elevators	$y_e = 40/17$
For stacking conveyors	$z_e = 12/17$

The sum of $x_e + y_e + z_e = 35/34 + 80/34 + 24/34 = 139/34$, or $4\,3/34$ total equalized profit groups.

The second set of equations, which includes the Erection Shop, is:

$$.08x_e + .30y_e + .30z_e = 1.000 \text{ Foundry}$$

$$.32x_e + .225y_e + .20z_e = 1.000 \text{ Weld Shop}$$

$$.32x_e + .15y_e + .40z_e = 1.000 \text{ Erection Shop}$$

These equations solve to

$$x_e = {}^{25}\!/_{24}$$

$$y_e = {}^{20}\!/_{9}$$

$$z_e = {}^{5}\!/_{6}$$

The sum of $x_e + y_e + z_e = {}^{75}\!/_{72} + {}^{160}\!/_{72} + {}^{60}\!/_{72} = {}^{295}\!/_{72}$, or $4^{7}\!/_{72}$ total equalized profit groups.

The $4^{3}\!/_{34}$ for the Machine Shop is smaller than the $4^{7}\!/_{72}$ for the Erection Shop. This means that the Machine Shop's capacity is smaller than the Erection Shop's when these products are being made. So the Machine Shop is the third restriction.

The other way of determining which department is the third restriction is to check the "open" capacity in the departments and the requirements for shifting load between departments. This is similar to steps already described and will give the same answer as the simultaneous solutions.

11. Check for further profit potential

The same procedures are used to determine whether further increases in profit are possible. No further increase in profit is possible in this problem.

There is another way of checking the possibility of further profit potential. This is called the Northwest Corner Method. The procedure is as follows:

The sequences of the rows and columns are changed, but each number is kept in its original row and column. Such an arrangement is shown in Table 6-7. The first key number is put in the upper left-hand, or northwest, corner of the matrix. This immediately establishes the restriction for the first row and the product for the first column. The second key number is placed one row down and one column to the right. Then it is possible to fill in four entries in the northwest corner of the matrix. In the example the northwest corner entries are as follows:

.300	.080	—	—	—	—	= 1.000
.225	.320	—	—	—	—	= 1.000

These values are the same as the coefficients shown in Table 6-5, Unitized, Equalized Matrix, given on page 135. There is one requirement to

Table 6-7. Northwest-corner System for Finding Determinant in a Matrix

	Pallet elevators	Fork lift trucks	Stacking conveyors	Traveling cranes	Jib cranes	
Foundry	.300	.080	.300	.400	.200	= 1.000
Weld Shop	.225	.320	.200	.300	.400	= 1.000
Machine Shop	.200	240	400	.300	.100	= 1.000
Forge Shop	.150	.120	400	.200	.200	
Finishing	.200	.200	.400	200	.300	
Erection Shop	.150	.320	.400	.150	.200	

be met in the northwest-corner arrangement, and that is that the numbers to the right of each key number must not be larger than the key number.

The next step in the alternate-checking procedure is to see whether any other key number can be added to this group of figures. The stacking conveyors meet the requirement. The stacking conveyor .300 in the Foundry is not larger than the Foundry Row Key Number. The .200 in the Weld Shop is smaller than the Key Number .320 for the Weld Shop. It is still necessary to select the Third Key Row or department. This can be done as it was done before.

At this point there are no other products that do not have a larger number than the key number in one of the rows. The traveling cranes take too much time in the Foundry and the jib cranes take too much time in the Weld Shop. The final simultaneous equations are:

$$.30y_e + .08x_e + .30z_e = 1.000$$

$$.225y_e + .32x_e + .20z_e = 1.000$$

$$.20y_e + .24x_e + .40z_e - 1.000$$

These equations are the same as those developed by the other method.

12. Set up a final program and profit picture

This step is the interpretation of the results of the previous steps and the development of a final program and profit table, as shown in Table 6-8. This would be used as a guide for management decisions on the company's manufacturing activities. The equalized profit groups x_e, y_e, and z_e of fork lift trucks, pallet elevators, and stacking conveyors for the optimum program can be added together. This gives $4\frac{3}{34}$ total groups with a total profit of $4\frac{3}{34} \times \$6,000$, which equals $\$24,529\frac{7}{17}$.

These three equalized groups can be converted back to actual numbers

Table 6-8. Final Program and Profit Table

Capacity			Production hours used per month				
Departments	Hours available	Hours used	Traveling cranes	Jib cranes	Fork lift trucks	Pallet elevators	Stacking conveyors
Foundry	2,000	2,000	—	—	165	1,412	423
Forge Shop	1,000	759	—	—	124	353	282
Weld Shop	1,000	1,000	—	—	330	529	141
Machine Shop	8,000	8,000	—	—	1,980	3,770	2,250
Finishing	2,000	1,917	—	—	413	940	564
Erection Shop	6,000	5,790			1,980	2,120	1,690
Equalized profit units			-	—	$35/34$	$40/17$	$12/17$
Actual units produced			—	-	$8\ 4/17$	$11\,13/17$	$14\ 2/17$
Profit per actual unit (in dollars)			6,000	300	750	1,200	300
Profit (in dollars per month)			—	—	$6{,}176\,8/17$	$14{,}117\,11/17$	$4{,}235\,5/17$
Total profit per month (in dollars)					$24{,}529\,7/17$		

of units by multiplying by 8, 5, and 20 respectively. These are the same multipliers used when equalizing the columns in the ratio-analysis matrix, and they convert x_c back to x for fork lift trucks and similarly for the other products.

Knowing the actual units to be produced makes it possible to calculate the actual hours used in each of the production shops and the profit dollars per month for each product. These values have already been calculated indirectly by using the equalized groups, but for convenience of tabulating the result they are shown in Table 6-8. The total profit from this optimum program, calculated previously to be $24,529, is checked by totaling the profits from the three chosen products. Table 6-8 indicates that the maximum profit is obtained by producing:

$8\,4/17$ fork lift trucks per month
$11\,13/17$ pallet elevators per month
$14\,2/17$ stacking conveyors per month

It also indicates that the capacity of the Foundry, Weld, and Machine Shops is completely utilized and that other shops have unused capacity. The solution gives complete operating and profit information for management use.

SOME COMPARISONS OF RATIO ANALYSIS AND SIMPLEX

There will be cases where it is impractical or perhaps impossible to solve the matrix by ratio analysis. In these cases it will be necessary to use the simplex method or some other procedure. The ratio-analysis method, however, can be used to minimize the number of iterations in the simplex method. Then the starting column for the simplex method is selected in the same way as the ratio-analysis method. If necessary the ratio-analysis method can be used to select the key columns for the first three or four iterations. Quite frequently this will save calculation time because of the method of selecting the starting point in each method. The ratio-analysis method starts with the benefit that will bring the largest over-all returns; the simplex method starts with the benefit that brings the largest unit profit.

In the example above the simplex method would start with traveling cranes, the largest *unit* profit. In the ratio-analysis solution, the pallet elevators, the largest *over-all* profit for a single product, provided the starting point. The starting point selected by the ratio-analysis method will save an iteration when this problem is solved by the simplex method; because the regular simplex method will bring in the traveling cranes and then drop them. Doing this requires an iteration of the matrix which is not needed when we start with the most profitable item.

The simplex method can include secondary limitations, such as "At least 30 of Product B must be made," "Not more than 30 of Product K can be made," and so on. In the ratio-analysis method all minimum requirements, such as "30 of Product B must be made," are handled in the following way: the amount of the resources necessary to make these minimum requirements is calculated and totaled, and this amount of each resource is deducted from the available amount of the resource before unitization takes place.

Any column can be dropped from the simplex matrix when the following conditions exist in the corresponding ratio-analysis matrix:

1. Relationship between two columns is such that each number in one column is equal to or *larger* than the number in the same row of the other column, with at least one number in the column to be dropped *larger* than the corresponding number in the other column.

2. There are no outside quantity restrictions on the amount of the demand in the column with the smaller figures.

There may be requirements for a specified amount of the item which is to be dropped from the matrix. This does not change the situation, because the capacity to produce this specified quantity can be deducted from the total capacity before the matrix is unitized.

In actual practice, although not in theory, it may be possible in the ratio-analysis matrix to drop any column with many large numbers and no particularly small numbers.

It is possible to drop any row from the simplex matrix when the following condition is met:

The relationship between two rows is such that each number in one row is equal to or *smaller* than the *corresponding number in the other* row, with at least one number *smaller* than the number in the other row. The row with the smaller numbers will be dropped.

Such a row will be the last limitation to come into use and will only be used to bring one more benefit into the group, if any spare capacity is available.

SUMMARY

The ratio-analysis method presents a way of solving some problems and of minimizing the work necessary to solve other problems.

Where it is possible to select the bottlenecks or restrictions, the problem can be solved as a simple set of linear equations. Ratio analysis provides a way of picking out these restrictions in some cases. Where it applies, it is a rather simple process. It amounts to a procedure for selecting restrictions and a corresponding group of simultaneous equations out of a matrix constructed from inequations.

A summary of the steps used in the ratio-analysis method follows:

1. State the problem in terms of the objective.
2. Collect the information relating to the problem.
3. Form a ratio-analysis matrix as follows.
   a. Find a common unit of measure for resources and demands.
   b. Establish the unknowns.
   c. Set up the information in inequation form.
4. Deduct the capacity required for a specified amount of an item that must be in the program.
5. Unitize the rows of the ratio-analysis matrix.
6. Equalize the columns of the ratio-analysis matrix.
7. Find the first key number.
8. Determine whether improvement is possible from combinations and the amount of the improvement.
9. Determine whether further improvement is possible from larger combinations.
10. Determine further limiting resources.
11. Check for further improvement.
12. Set up and evaluate the final program.

As shown by the sample problem solution, the ratio-analysis method of linear programming can be used to aid management in making de-

cisions. It is a formal, logical approach to making a best decision or choice where alternatives and choices exist. The procedure will find an optimum solution from a number of alternatives and a mass of information which would be impossible to handle by normal analysis or calculation.

The ratio-analysis method of linear programming aids in understanding the simplex method. It shows that what is being done in the simplex method is to find a set of simultaneous equations that are in the matrix.

CHAPTER 7

The Index Method

The first of the approximate methods to be discussed is the Index Method. In a strict mathematical sense, the index method is not an LP method because it does not indicate precisely when the best answer has been reached. Nonetheless, it is a programming procedure in the sense that it provides a first answer and, by the use of index numbers, permits the development of improved solutions by the judicious use of alternatives to overcome restrictions and limitations.

In this section we shall discuss the following:

1. The background of the development of the index method
2. The underlying principles and concepts
3. The general procedure and rules for using the method
4. The application of the method to a machine-loading problem
5. The programming principles of the index method

BACKGROUND AND DEVELOPMENT OF THE INDEX METHOD

The index method was developed to solve a scheduling and loading problem involving a large number of orders and automatic screw machines.[1] From a survey it had been established that the automatic department was the bottleneck or restricting department that limited the amount of production and profit that could be obtained from the plant. To put it another way, the plant had sufficient facilities in the other departments to more than process all of the production that could be machined in the automatic department. The limiting or restricting resource was the automatic screw machines themselves. A study of the machines and the orders to be run showed that within the department only a few machine groups were restricting or limiting production. As a matter of fact some of the machine groups or centers were not being fully utilized and had not been for some time.

The situation was complicated further because schedules were pre-

pared weekly and changed frequently as the result of sales-department requests, unexpected equipment failure, absenteeism, and the like. These frequent changes made it impractical to use the simplex method for routine planning. Recomputing answers that reflected the changed conditions would have required too much time. Necessity was the mother of invention. The index method was conceived and developed by working back from the simplex to obtain a practical and useful tool to handle this problem.

The method turned out to be fairly simple, routine, and easily taught. Because of the simplicity and rapidity with which the calculations could be carried out, the index method was installed for loading machine centers. Results of 3 years of experience to date indicate a remarkable improvement over previous pencil-and-paper methods,[2] closer control of production, and greater ease of management.

Since the index method was developed to solve a scheduling and loading problem, we shall use scheduling and loading as a background upon which to develop the basic concepts and principles of the method.

A scheduling and loading problem as seen through programming eyes can be solved for least time, least cost, highest utilization of equipment, or some other objective. In order to demonstrate the method and the linear-programming concepts, a problem of loading machine centers in *least time* to meet delivery promises will be used.

BASIC CONCEPTS AND PRINCIPLES UNDERLYING THE METHOD

A look at several definitions will facilitate understanding the method. The definitions that we want firmly in mind are as follows:

Ideal Machine is the machine that will perform the operation on a part in the least amount of time, or at least cost, or at most profit, or in an optimum fashion in terms of some other objective. It is the best machine for the part or order in terms of the objective sought.

Index Number is a number that represents the extra or additional time taken (or cost incurred if that is the objective) when a part or order is produced on other than the ideal machine. It is a number that indicates that a "penalty" is involved when other than the ideal machine is used. An index number is computed only where there are *alternative* machines for performing the operation. The rule for computing index numbers is given by the following formula:

$$\text{Index number} = \frac{\text{alternate machine time} - \text{ideal machine time}}{\text{ideal machine time}}$$

Index Method is a procedure for computing indexes for each order or operation and assigning the *overload* from ideal machines to alternative machines using the index number to minimize the excess time (or cost or the like) taken to produce the order. Basically, it is a procedure or method which indicates the extent to which non-ideal machines cost more in time, money, profits, or the like, than ideal machines.

The computation of an index number is a straightforward process. For example, assume that an order for 400 pieces of Part W321 which is scheduled for next week can be run on Machine Groups 61, 62, 63, and 64 at different production rates. When the time required to produce the 400 pieces on each of the machine groups is computed, we have order times as shown in Table 7-1.

Table 7-1. Order Times for Part W321

(In hours)

Machine Group 61 *	Machine Group 62	Machine Group 63	Machine Group 64
30	34	60	75

* Ideal least time for the order.

By use of the formula the index numbers are as follows:

$$\text{Index number (Machine Group 61)} = 0 \text{ (ideal)}$$

$$\text{Index number (Machine Group 62)} = \frac{34 - 30}{30} = \frac{4}{30} = .13$$

$$\text{Index number (Machine Group 63)} = \frac{60 - 30}{30} = \frac{30}{30} = 1.00$$

$$\text{Index number (Machine Group 64)} = \frac{75 - 30}{30} = \frac{45}{30} = 1.50$$

The index numbers for each order to be scheduled in a given period are computed in the same way. The index numbers once computed are directly comparable for scheduling and loading purposes even though derived from different orders. The index number represents a percentage increase in time resulting from using other than ideal machines. In these sample calculations, the index number represents the percentage increase in hours required to run an order on a non-ideal machine.

It has proved useful in practice to set up the information about the order or part on a card, which is filed according to the week or period in which the part is to be run. The deck of cards for a given period then provides, in a convenient form, the scheduling information for planning the load to machines.

A sample card which sets forth the information needed and indicates the calculations required for the index method–time solution is shown in Table 7-2.

Table 7-2. Schedule Card

Part no	Operator no	Tooling	Material	Pieces on order	Pieces scheduled	Alternate machine	Alternate machine	Alternate machine	Alternate machine	Alternate machine
A3470	65	✓	✓	3,500	3,500					
Machine group						Type I		Type II		Type III
Time per piece (in hours)						0240		0286		.1080
Running time (in hours)						84		100		378
Setup time (in hours)						6		8		5
Total time (in hours)						90		108		383
Cost factor (machine hour costs)										•
Equalized costs										
Index number--time						000		20		3.26

GENERAL PROCEDURE AND RULES FOR USING THE INDEX METHOD

The general steps for applying the index method to scheduling and loading problems are as follows:

1. Set up a scheduling table or matrix for working out the machine load.

2. List orders to be run in the period—including carry-overs from present period.

3. Determine alternate machines on which each order can be run.

4. Compute time to run each order on alternate machines.

5. Calculate an index number for each order on each machine using the ideal machine as a base.

6. Assign orders to machines on which they can be run. Circle the ideal.

7. Total the ideal assignments (circled values) to each machine.

8. Transfer overloads, first selecting orders with smallest index number and proceeding to higher-index-number orders until a satisfactory balance is reached.

STEPS FOR SETTING UP AND SOLVING A MACHINE-LOADING PROBLEM FOR LEAST TIME

Now that we have covered the basic concepts and principles and listed the rules and procedures for using the method, let us use them to solve a small machine-loading problem.

1. Loading machine centers for least time

The problem and the objective is to assign a group of orders to machine centers in the least amount of time. Do not create additional setups by splitting orders. Information about the orders, the quantity to be run, the various automatic machine groups, running time per order by machine, and hours available are given in Table 7-3.

Table 7-3. Order and Production Information

Part no.	Stock size	Quantity	Machine groups in order of inc. stock size				
			No. 61 (2)	No. 62 (1)	No 63 (4)	No. 64 (5)	No. 65 (2)
W321	.985	400	30	34	60	75	—
W326	2 125	600	—	—	—	90	60 *
W330	1.050	800	60 *	140	210	150	—
V200	1.050	400	90	135	112	115	160
V210	1 500	500	-	—	160	188	210
V220	1.225	1000	—	220	70	130	115
Hours available			60	35	185	190	60

* Running in the present schedule on the machine group indicated.

The hours available have been adjusted by a utilization factor that includes down time, maintenance, repair time, and absenteeism. Production times include setup time.

Orders W326 and W330 are running on their ideal machines in the present schedule. Therefore, they must be carried over to the schedule being developed for next week. By that time, Order W326 will require 600 pieces and Order W330 will require 800 pieces to complete.

Applying the rules for solving a problem by the index method to the information as given results in a starting matrix, as shown in Table 7-4.

Table 7-5 shows the result of carrying out steps 1 to 8.

Table 7-4. First Schedule—Third Week

	Machine Group 61	Machine Group 62	Machine Group 63	Machine Group 64	Machine Group 65	Excess hours
W321	.00 30	.13 34	1 00 60	1 50 75		
W330	.00 60	1.33 140	2.50 210	1 50 150		
V200	.00 90	.50 135	.24 112	.28 115	.78 160	
V220		2.14 220	.00 70	.86 130	.50 115	
V210			.00 160	.18 188	.31 210	
W326				.50 90	* .00 60	
Available hours	60	35	185	190	60	
Ideal hours	180	0	230	0	60	
Difference	+120	−35	+45	−190	0	

* Order set up on the machine center indicated

In Table 7-4 Machine Groups 61 and 63 are overloaded. They restrict or limit production for the period—if all orders are to be run on ideal machines.

There are *alternate* machine groups on which the orders can be run so that the bottleneck or restricting condition can be eliminated. The problem then is to transfer the overload with a minimum of excess hours over ideal.

The index numbers have been computed to serve as a guide for shifting overloads at a minimum of excess hours. The rule is to move orders from overloaded to underloaded machines in order of lowest index numbers to higher index numbers.

Orders W321, V200, and V210 have the lowest index numbers for alternate machines and should therefore be considered for transfer first. In general, these orders can "buy," time on underloaded machines at the cheapest price—the lowest index possible. Order V220 should be then considered. Orders W330 and W326 should remain as loaded be-

Table 7-5. Least-time Schedule—Third Week

Order	Machine center					Excess hours
	Group 61	Group 62	Group 63	Group 64	Group 65	
W321	30 .00	34 .13	60 1.00	75 1.50		4
W330	60 .00	140 1.33	210 2.50	150 1.50		
V200	90 .00	135 .50	112 .24	115 .28	160 .78	22
V220		220 2.14	70 .00	130 .86	115 .50	
V210			160 .00	188 .18	210 .31	28
W326				90 .50	60 .00	
Available hours	60	35	185	190	60	
Scheduled hours	60	34	182	188	60	54

cause they are already set up on ideal machines and use the available time in the period. In general, they are properly assigned because their index numbers are higher on alternate machines than the remaining orders anyway.

This program will meet the schedule requirements for the week. All orders are assigned to machine groups, and there are very few unutilized machine hours. None of the orders have been split, and thus additional setup time is avoided.

Orders W321, V200, and V210 are transferred to non-ideal machines at a penalty of 54 excess hours. The index method, in addition to being a useful production-planning procedure, provides valuable by-product information. In practice, it has been possible to tabulate the overload on ideal machines as a guide to the purchase of new equipment and manning arrangements. It also provides useful information for evaluating the worth of acquiring additional tooling, and the desirability of splitting orders.

If the conditions of the problem just solved allowed splitting of orders, the least-time schedule given in Table 7-6 would have resulted. This

Table 7-6. Least-time Schedule Splitting Orders—Third Week

Order	Machine center					Excess hours
	Group 61	Group 62	Group 63	Group 64	Group 65	
W321	30 $\boxed{.00}$	34 $\boxed{.13}$	60 $\boxed{1.00}$	75 $\boxed{1.50}$		4
W330	60 $\boxed{00}$	140 $\boxed{1.33}$	210 $\boxed{2\ 50}$	150 $\boxed{1\ 50}$		
V200	90 $\boxed{.00}$	170 $\boxed{.50}$	112 $\boxed{.21}$	115 $\boxed{.28}$	160 $\boxed{.78}$	25
V220		220 $\boxed{2\ 14}$	70 $\boxed{00}$	130 $\boxed{.86}$	115 $\boxed{.50}$	
V210			115 160 $\boxed{.00}$	53.1 188 $\boxed{.18}$	210 $\boxed{.31}$	8.1
W326				90 $\boxed{50}$	60 $\boxed{.00}$	
Available hours	60	35	185	190	60	
Scheduled hours	60	34	185	168.1	60	37.1
Difference	0	1	0	21.9	0	

program has 16.9 *fewer* excess hours than the preceding least-time schedule (54 − 37.1 = 16.9). It requires one more setup because Order V210 is split, but it also makes available 21.9 hours on Group 64 for producing other orders. Either the reduction in excess hours or the addition of available hours for other orders can be evaluated in dollars. The cost of additional setup can be compared to the volume and a decision to split an order or not made on the basis of the dollars and cents involved.

2. Loading machine centers for least cost

The procedure for calculating a schedule resulting in least cost involves calculating a cost factor and equalized costs before computing

the index numbers. From that point on the basic procedure is the same as used in the time solution—to schedule by lowest index numbers when overloads occur. Table 7-7 shows an example of the calculations as required for least-cost solution.

Table 7-7. Schedule Card

Part no.	Operator no.	Tooling	Material	Pieces on order	Pieces scheduled	Alternate machine	Alternate machine	Alternate machine	Alternate machine	Alternate machine
A3470	65			3,500	3,500					
Machine group						Type I		Type II		Type III
Time per piece (in hours)						.0240		.0286		.1080
Running time (in hours)						84		100		378
Setup time (in hours)						6		8		5
Total time (in hours)						90		108		383
Cost factor (machine hour costs)						6		4.5		4
Equalized costs						540		486		1,532
Index number—cost						.11		.00 *		2.15

* On a cost basis, the Type II is the best machine, whereas on a time basis, the Type I is the best.

The Cost Factor for a machine group is the per hour operating cost expressed in dollars. (Per hour operating costs generally consist of direct labor and other assignable costs.) Equalized Costs are calculated for each alternative machine and are the product of total time and the cost factor. They are used to compute index numbers for a cost solution. An example of the cost calculations for a given order is shown in Table 7-7. The programming procedure for assigning orders to machine center at minimum cost is given in detail in Table 7-8.

LP PRINCIPLES IN THE INDEX METHOD

The index method is an effective method that has passed the trials of practical application. It was conceived and developed on programming principles, and by itself it is a valuable method for loading machine centers. In practice, it has achieved considerable savings and improved deliveries to customers.

The programming ideas inherent in the use of the index method are as follows:

1. The *objective* must be stated—least time, least cost, or the like.
2. The *restrictions* on the objective must be overcome or alleviated by the existence of alternatives.

Table 7-8 [*] The Programming Procedure for Assigning Orders to Machines to Minimize Costs

NEW SHOP ORDERS

Make a 4 × 6 white card for each new shop order to be scheduled. Card shows:

Shop Order No	Machine Code
Part No	Running Time
Stock O D	Set-up Time
Date Required	Total Machine Hours

Compute the total machining time if job is done on the ideal machine code (the group of machines that will do the job in the fewest number of hours). Take time data from record file—add full set up time to running time for that size order.

EXISTING ORDERS

Make a 4 × 6 buff card for each existing order which is already scheduled or being machined. Card contains information similar to white cards.

Compute the remaining machining time to complete the order.

↓

NOTE: FOR BEST RESULTS RE-SHUFFLE ENTIRE SCHEDULE EACH WEEK

Sort cards representing all orders to be scheduled during period into three groups: inner races, outer races and miscellaneous parts. This will make it easier to combine orders requiring the same set up.

↓

1. Arrange each group in order of increasing bar stock sizes
2. Group like orders (those requiring the same set up) → Make one composite card for each group of like orders. Compute new time to machine composite order with only one set up

↓

Group orders according to ideal machine code for running the job ←

↓

Total the hours to be scheduled on each ideal machine code

↓

Determine the machining hours available in each code for the scheduling period. Use information from shop on number of machines, downtime and total working time available → Compare the total hours to be scheduled on each machine code with the available hours for machining in that code → Codes with scheduled hours less than available hours are not overloaded. These cards are laid aside temporarily

↓

Codes where hours to be scheduled are more than available hours are overloaded. These cards are handled as follows:

↓

1. Compute total time to machine each order on alternate machine codes where job can be performed. Total time equals running time for order plus full set up time
2. Multiply total hours by cost factor to obtain equalized hours for each alternate code

↓

1. Compute excess hours for each alternate code for each job. (Excess hours is alternate hours minus Ideal hours.)
2. Divide excess hours by Ideal hours to obtain an index number for each alternate. For example:

$$\frac{100 \text{ excess hours}}{400 \text{ ideal hours}} = 25 \text{ Index number}$$

$$\frac{200 \text{ excess hours}}{400 \text{ ideal hours}} = 50 \text{ Index number}$$

↓

Shift orders from overloaded codes to codes that are not overloaded. Move first to codes with lowest Index numbers in order to minimize excess hours. Additional shifts may be made in order of increasing Index numbers. Look for ways to shift orders in double and triple shifts to obtain a schedule pattern that takes advantage of the most economical production facilities ←

↓

1. Regroup all order cards according to most desirable codes for machine scheduling
2. Total the hours of machine loading for each machine code → Compute the excess hours assigned to each code. Make up a summary sheet showing:
   1. Order No
   2. Bearing No
   3. Part No
   4. Ideal Machine Code
   5. Ideal Time
   6. Assigned Code
   7. Assigned Time
   8. Excess Hours

   ↓

   Prepare separate reports on the reasons for excess hours

↓

Hang each group of cards on a separate hook, one for each machine code

↓

Schedule orders from these card groups to maintain the delivery schedule required

[*] Reproduced through the courtesy of *Factory Management and Maintenance and American Machinist*

3. The orders or *demands compete* with each other for use of a *fixed* amount of machine time or *resources.*

4. The method involves a number of *specific computational rules* and procedures for establishing a first program and improving it.

5. The method requires an *orderly arrangement* of information which, for ease of computation, is set forth in a table or matrix.

6. The concept of *interdependence* is present only in the sense that a change in the time to run an order will effect the total time taken. The change in the time to run any one order introduced by shifting may or may not affect the time to run another order—depending on whether double or triple shifts are involved.

7. Excess hours are a measure of the additional or *opportunity time* (*cost*) taken in selecting one program over another.

8. The concept of linearity, although not as evident in the index method as in some of the other concepts, is nonetheless present. An increase in the number of machine hours used will bring a proportional increase in output.

The index method does not contain the following linear-programming concept: a *specific optimum* or best answer which is clearly designated when it has been calculated.

SECTION THREE

Application

Experience gained in the application of LP methods to management problems indicates that successful installations generally follow a three-step pattern. The first step consists of recognizing that the problem is one that can be solved by LP methods. The second step consists of making a trial installation on a pilot or model basis to determine the potential before committing time and effort to full-scale application. The third step consists of working out the solution to the problem selected and then interpreting the solution into a workable program for people to follow.

The chapters in this section are arranged to correspond to the pattern of application just described. First comes Chapter 8, entitled "Recognizing Problems Which Can Be Solved by Linear-programming Methods." The basic kinds of LP problems are identified and discussed and methods suggested for solving each type. Also included in the chapter are rules for setting up data and information in usable form and guides for determining when the necessary information has been obtained.

Next comes Chapter 9, entitled "The Model," which presents the way in which various problem areas can be explored for potential improvement. With this information provided, management can direct application effort to areas that will bring the largest return for the effort. It also prevents the situation in which the problem was recognized and solved only to find that the improvement was not worth the cost of getting it.

Then come three chapters, each one of which describes an industrial application that has been working and is continuing to work successfully in practice. In general, the applications are aimed at showing how LP methods provide information for production planning, profit planning, and sales planning.

The first application is given in Chapter 10, entitled "Maximizing Profit Margin Considering Manufacturing and Distribution Costs." This application shows how headquarters planning used LP as a system for assigning production to outlying plants. Differences in operating costs and efficiencies in each plant, as well as freight equalization and the

market to be served, were taken into account in arriving at a best solution for planning purposes.

In the application given in Chapter 11, entitled "Obtaining the Most Profitable Share of Your Market," LP was used to direct sales efforts and develop sales programs for using plant capacity as effectively as possible under forecast conditions.

In Chapter 12, entitled "Stabilizing Production and Employment Levels at Least Cost," LP was used to integrate purchase-make, inventory, and forecast information into a least-cost program for operating a plant at a stable level of employment.

Chapter 13, entitled "Putting Linear Programming to Work: Problems, Possibilities, Predictions," attempts to answer a number of questions which arise in connection with putting linear programming to use in a firm. Such questions as: "Where does it fit in the organization?" "Who should do the work?" and "What personal qualifications are desirable?' are discussed. Also presented are some thoughts on the trend of linear programming in the future.

This section has several objectives: (1) to demonstrate further the use of the methods described in the methods section, (2) to present problems that demonstrate the kind of information needed and the level of result obtained, (3) to provide the reader with an opportunity for developing skill and proficiency in application, and (4) to suggest what lies ahead.

CHAPTER 8

Recognizing Problems Which Can Be Solved by Linear-programming Methods

A wide variety of management problems have been solved by conventional linear-programming methods. In addition, practical simplifications and combinations of the conventional methods have resulted in successful solutions to many other problems.

The versatility of the methods and approach offers considerable promise for solving a great many problems better than heretofore. This can prove to be a booby trap, because recognizing and setting up a problem as one which can be solved by linear-programming methods (either conventional or approximate) is not easy. As a matter of fact, in some instances it is extremely difficult, if past experience is any guide. Accordingly, the discussion which follows is intended as a guide for recognizing and identifying a problem as one that can be solved by linear-programming methods. Recognizing that a problem is *not* a programming problem is just as important as recognizing a potentially successful application.

BASIC KINDS OF LP PROBLEMS

In recognizing linear-programming-type problems it is necessary to remember that as a method it is an "allocation" or "assignment" technique. Fundamentally, the computational process allocates or assigns a limited amount of company resources, such as capacity, among competing demands, such as customers' orders, according to an established objective, such as least cost. The *relationship* of demands to resources helps to define or categorize the basic kinds or types of linear-programming problems *and indicates the most effective method.*

The first type of problem is one in which *demands exceed resources.* For sake of reference we call this Type 1. In this kind of problem, all demands cannot be met for the period or conditions being considered. Consequently the problem is to select the demands that use the resources in the optimum, or best, manner.

Type 1 problems are solved by using the simplex and ratio-analysis methods. With the selection of a common unit of measure and the addition of an artificial or dummy resource, the modi method can be used under some circumstances. If the problem is not too large, the ratio-analysis method will prove to be the quickest of the methods.

Type 2 problems are those in which *demands equal resources.* In this general kind of problem, all demands can be satisfied if all resources are used. The problem therefore is to allocate or use all resources to best advantage in satisfying all demands. Type 2 problems usually are best solved by the modi method where it is possible and otherwise by the simplex method where it is not. When conditions make it possible, the modi is especially effective in problems of this type because advantage can be taken of its timesaving and practical features.

Type 3 problems are problems in which *resources exceed demands.* In problems of this type the prime consideration is to make the most economical assignment of demands to resources. Frequently it happens that there are a few limiting conditions even though the resources as a total exceed demands. For example, in some machine-loading problems, particular machine groups may be overloaded during a period, whereas all machine groups together have idle capacity. In this kind of Type 3 problem, the index method has proved to be effective. In Type 3 problems of considerable size in which the relationships tend to be obscure, the modi has proved to be effective.

Irrespective of the type or category into which a particular problem falls, it is necessary to have a clear statement of the problem and the objective, the correct unit of measure, and the most effective method of solution. Certain questions, if used as a guide, will aid in recognizing and categorizing the problem and indicate the most effective method. These questions represent a step-by-step routine, which, when followed, will lead to a conclusion about the possibilities of using linear-programming methods.

It is necessary to point out that this is not a foolproof procedure. It is a guide that has proved valuable from practical experience. It provides a way to begin and proceed to reasonably sound conclusions regarding the potential and usefulness of linear programming for the problem being considered.

1. Is the problem a linear-programming-type problem?

Does it fall into one of the basic types in which the desired solution consists of a number of related, interdependent decisions for assigning the resources to demands? It is a programming problem when it involves situations and conditions of the following kind:

The best assignment of orders to a number of plants in which they

can be produced can be decided only after a number of decisions or assignments have been considered, because the assignment of an order to capacity in one plant prevents the capacity from being used for some other order. In other words, we must have alternate courses of action to choose from, and we must have a specific limit on our resources or demands.

There may not be a problem. Management may have no difficulty making the best decision or arriving at a rapid solution to complex problems.

On the other hand, if there is a problem, it is necessary to know whether the problem is a linear-programming-type problem and if so what the solution is required to show. This brings us to the second question.

2. What is the objective or goal?

Now that the basic character of the problem has been established, we need to know the kind of solution desired. In other words, can the problem be *stated* so that a desired objective, such as least cost or greatest quantity, can be established clearly? We need to know what the solution is required to show in terms of objective or goal. If there is a potential linear-programming problem, then *single objectives* of the following type can be stated:

1. What is the *most profitable product mix* for our products?
2. What is the *least-cost assignment* of orders to machine centers to meet delivery promises?
3. What assignment of production releases to individual plants will *maximize gross-profit margin over all?*
4. What items should be carried in inventory to meet seasonal fluctuations in sales and production at *least inventory carrying cost?*

The problem will not be one that can be solved by linear-programming methods if more than one objective is desired at a time or if statements of the following type are all that can be made:

1. The plant must be operated more efficiently
2. Costs must come down.
3. We want better delivery to customers.

It is one thing to say we want more production so that delivery to customers can be improved. It is quite another to say we want the *least-cost program* that will enable us to meet delivery promises. The point to be made is that the problem needs to be *clearly expressed in terms of the objective* before a start can be made. The problem may be restated before the solution is attempted and may be completely changed before the final answer is obtained. But the start must be made, and getting agreement and deciding on an objective is the place to start. Do not be

surprised, however, if a different objective becomes more worthwhile by the time the solution is reached. This is one of the advantages of the linear-programming approach. It forces the problem and objective into perspective and frequently brings about a realignment of thinking about problems and situations that were not possible before simply on the basis of comparisons.

3. What are the restrictions or limiting conditions that prevent the objective from being attained?

The next step is to determine the restrictions and conditions that limit or reduce the possibility of attaining the objective. There are usually a number of these in most industrial situations. Many of them will fall into the following general categories:

1. Capacity restrictions—only a limited number of hours are available on milling machines, looms, assembly lines, and the like for the schedule period.
2. Man-power restrictions—a certain number of highly skilled cold-mill rollers limit the number of shifts that the temper mills will operate.
3. Material restrictions—there may be only so many pounds of brass available this month for tubing and rod orders.
4. Space restrictions—only a limited amount of cubic feet of warehouse and storage space are available before the cost of additional space becomes more than the value of the parts to be stored.
5. Sales restrictions—the sales department requests a minimum of 10 and a maximum of 14 tank cars of naphthalene this month.
6. Capital restrictions—the board of directors has stated that only $15,000,000 will be allowed for additional raw material and new equipment this year.

Generally, if there *are* restrictions or limitations of the kind just mentioned, they indicate a potentially successful linear-programming application.

If there are *no* restrictions or limitations there is usually no linear-programming problem. This is likely to happen under the following conditions:

1. There are available all the centerless grinders that are needed for producing drill rod at least cost.
2. All of the automobile glass required can be produced in the plant which has the cheapest transportation costs.

Recognizing and stating the restrictions or limitations is a key step. The restrictions limit the amount of profit or number of pieces that can be obtained. The value of linear programming lies in the way in which it overcomes the restriction by using the alternatives to advantage. This

is what is meant when we say that linear programming is a method or technique for allocating a firm's limited resources (machines, money, time, people, or material) among the various factors (products or customers) requiring their use according to some criterion (profits, cost, or quantity).

There have been a number of cases in which restrictions had to be put into the problem artificially in order to use linear programming. For example, several production-planning problems have been solved with better results than were obtained by the usual methods by including fictitious products which forced the real products to be produced in the best way. This is strictly a computational device needed to satisfy the mathematical requirements. The technique does not discriminate between real and fictitious numbers and products.

4. Are there alternative ways which will lead to overcoming the disadvantages of the restrictions and accomplishing the objective?

Without alternatives there can be no linear-programming solution. There is no linear-programming solution if the following conditions are true:

1. Action must be taken one way or not at all.
2. There are only two alternatives. Generally two choices can be evaluated rapidly and compared for relative advantage and the most beneficial decision made without the use of linear programming.

There is a potentially successful linear-programming solution if the following condition holds:

There are more than two alternatives. In most business problems of any size, the choice of the best course of action is usually difficult to determine accurately without the use of linear-programming methods, which take into account all the interrelated alternatives and restrictions.

Alternatives must be *real* alternatives—that is, selecting one of several *different* choices gives a result which is *different* from results of selecting one of the other alternatives. For example, if you can *buy* a part for $10 and sell it for $12—and an "alternative" is to pay $8 for material, process it at a cost of $2 with the same selling price of $12—there is no real alternative for a maximum-profit program. *All the other things being equal,* the profit is $2 in both cases.

However, in this example, if the $2 processing keeps some men busy who would otherwise have to be paid for being idle, we have a real alternative. The advantage of having work for these men can be specifically stated in dollars. If the men had to be laid off and, later, men had to be rehired and trained, this cost disadvantage would have to be assessed in dollars in order to use it in the problem.

Similarly, if the quality of the self-processed article were higher, a real alternative exists and can be used in a linear-programming solution, if a specific value can be placed on this quality advantage.

Mathematically, the programming techniques use the various alternatives to overcome the restrictions and accomplish the objective. In short, linear programming makes the best choice in a situation where choices exist by reason of the alternatives.

5. Can the objective, restrictions, limitations, and conditions of the problem be stated numerically?

It is necessary for all the facts relating to the objectives, limitations, conditions, and alternatives to be stated numerically because linear-programming methods manipulate numbers. Ideally, they should be specific and accurate numerical values; but where certain information is not available, an estimate or approximation may be used. This estimate will give a valuable solution or optimum program, one which can be used until revision is possible with the actual figures in place of the estimate. This revision, or recalculation, will be very easy and rapid once the procedure has been set up using the original figures.

The facts must be stated numerically, either actual or estimated, to be used in a linear-programming solution. The numerical requirement is met as follows:

1. We cannot say, "It will mean layoff for 20 men." Instead we must estimate that the rehiring and training of 20 men will cost $2,000 as well as perhaps $10,000 of bad publicity (what is it worth to the managers to avoid the bad public relations resulting from the layoff?).

2. We cannot say, "These late deliveries or stockouts will cause us to lose customers." We must assess the value of a customer's patronage, the likelihood of losing him, and use a figure of, say, $10,000 per year (perhaps 5 possible lost customers at $2,000 per year each).

Frequently we can reverse this process and, by calculating two programs showing two specific alternatives, present a figure of what it will actually cost to avoid layoff for 20 men, or avoid late delivery and stockout.

Using this difference figure, management has facts on which to base a decision, to decide whether they are willing to take certain "risks" to gain a specific calculated benefit. They can truly say it is a "calculated risk" when they make a decision.

6. Are the variables interrelated?

The information and variables used in a linear-programming problem must be interrelated. Each one must be dependent on, and affected by,

other variables. This interrelation may be very obvious and direct or may be indirect and difficult to determine.

The profit dollars are obviously related to the cost of production and the selling price. Less obvious but interrelated in certain cases would be the humidity of the atmosphere, the amount of rejected material, or the total output. Either example would meet interrelation requirements, since if one variable changed the other variables would be affected in some way.

If we have several different processes which can each be carried out on only one piece of equipment, we would *not* have interrelation unless the end products were identical and there were limitations on these products for over-all demand. However, if these different processes can be carried out on more than one piece of equipment—that is, if they compete for the available time on the equipment—we have the required interrelation of variables by both equipment and product demand.

7. Can the objective, restrictions, and conditions be expressed mathematically?

For linear-programming application we must be able to express the objective, all restrictions, and pertinent facts as mathematical equations or inequations, and they must be linear, or capable of conversion to linear relationships. Linearity occurs when costs or profits go up or down proportionately with the level of production or other activity. This is the same straight-line or linear assumption that the break-even chart pictures.

"Better delivery results in improved customer relations" is *not* an equation. "Total profit = 10 pieces × \$10 profit per piece + 4 pieces × \$20 profit per piece = \$180" is an equation which can be used directly for a linear-programming solution. It is also *linear* and can be plotted as a straight line.

"Output of benzol this month must equal exactly 2,000 gallons" is an expression of a restriction as an equation. The statement "Output must be *less than or equal to* 2,000 gallons" is a restriction expressed as an inequation and represents a range of values within which output may vary and still satisfy the objective.

If the equations are *not* linear, it is sometimes possible, within the accuracy required for the problem, to convert to linearity by using one of the following procedures.

1. Use approximations for the relationship—perhaps a portion of the curve which is almost straight.

2. Use the "least-squares" method to obtain the mathematically best straight line to take the place of the curved one.

3. Split the problem into two or more parts where each part utilizes linear relationships or very nearly linear relationships.

If there is no possibility of converting to linear equations or other unsurmountable difficulties arise, it *may* be possible to:

4. Set up a formalized and simplified programming procedure to solve the problem. This may be merely a modification of the present calculation method, or it may be conversion to tabular procedures or a method using many shortcuts, but *we must be sure that the optimum* (or very nearly optimum) *program can be obtained* by the method used. The modi matrix, index, and ratio-analysis methods will be guides toward a procedure which will give an optimum answer.

5. Use nonlinear or quadratic programming. At the present time these procedures are in the process of development and in the future may be used for certain types of problems.

Investigating and answering the seven preceding questions represent the steps in the general procedure for recognizing problems which can be solved by linear-programming methods. The steps outline what has to be done in general and provide a way to get information and facts. The order or sequence can be changed to accommodate any given situation, and frequently the procedure involves working back and forth among several steps.

SETTING UP PROBLEM INFORMATION FOR USE BY LP METHODS

Assuming that the problem has been defined and expressed initially and that the necessary data and information are available, the next step is to set up the data in usable form for use by one of the available methods.

The selection of the method can have a considerable effect on the information and the work involved. In general there are a number of steps to be carried out in setting up the data before the method is finally established. The first step is to divide the problem information into manageable parts by partitioning vertically and horizontally.

1. Vertical partitioning of information

The purpose of vertical partitioning is to reduce the size of the problem if possible. This is generally done by looking for a bottleneck and then concentrating on it since it represents the critical part of the whole problem. A bottleneck might be a department or a machine group that limits the amount of production and profit that can be obtained in much the same way that the neck on a bottle limits how much can be emptied from or added to the bottle at any time. Concentrating on the bottleneck focuses attention on the critical area of a larger problem.

Finding the bottleneck can result in cutting down the problem to manageable size so that calculations can be carried out by hand and by

one of the specialized methods. If there is no single bottleneck but a number of them, then it may be necessary to use a computer if the problem is very large.

2. Horizontal partitioning of information

After a bottleneck is singled out by means of vertical partitioning, that bottleneck or area is then subjected to horizontal partitioning. The purpose of horizontal partitioning is not so much to reduce the size of the problem as to make it more convenient to establish a method and obtain an answer. Horizontal partitioning actually means dividing the problem into information areas. For example, in certain problems it may be necessary to obtain information about such items as the following:

1. Accounting—relating to costs and profits. The cost information may involve manufacturing costs by machine group, material costs, assignable-burden costs, distribution or freight costs, profit per piece according to the combination of machines used, and the like.
2. Manufacturing—relating to the process and process times. This information involves the steps in manufacturing (flow-process charts), the time per piece on the best and alternative machines, the effective capacity of equipment or machines, and the like.
3. Product—relating to the number and kind of product in the product line.
4. Sales—relating to customer demand, forecast information, minimum-sales levels, maximum expected sales, and the like.

In each of the basic areas all the variables and fixed factors should be identified and established. For example, in accounting, all variable and assignable costs need to be identified because they will be used in establishing the data used. On the other hand, the fixed costs are assumed to be constant and as such do not enter into the calculations. The list above is not intended to be comprehensive but merely suggestive of the information areas resulting from horizontal partitioning.

Horizontal partitioning of information also includes arranging all of the information in table form where it can be seen and reducing all information to either comparable or identical units. Quite frequently when this is done an insight into the problem is obtained that is as valuable as the answer itself.

3. Unit of measure

The unit of measure is a key point in selecting the method. If all data and information can be set up in identical units, such as standard machine hours, it then becomes possible to use the modi or transportation method. This simplifies the calculation and recalculation which is de-

sirable for staying abreast of changing conditions. If it is only possible to set up comparable units, such as individual pieces, time per piece, or individual capacities, instead of being able to relate them all to some standard, then the simplex method must be used. This method, as you know, is cumbersome, awkward, and time-consuming. As such it is not adaptable to problems which are required to be solved frequently. The point is that in setting up a unit of measure always strive for a standard or identical unit so that the simpler methods can be used.

In order to determine whether a problem can be solved by linear-programming methods. the following questions must be answered positively:

1. Is the problem a linear-programming-type problem? Does it fall into one of the basic types involving the relationship of demands to resources?

2. Can the problem be stated so that the objective desired is understood and set forth clearly? This is another way of asking whether we can state what we want to do.

3. Are there alternatives which can be used to obtain the objective? This question asks whether choices exist out of which a best program and best choice can be selected.

4. Are there restrictions or limitations to obtaining the objectives? This question is getting at information relating to capacities, such as machine hours, which limit how much can be produced.

5. Can the objective, restrictions, limitations, and conditions of the problem be expressed as linear equations and inequations? This question asks whether the problem is one that can be solved by linear programming. It provides the point at which method can be considered and frequently gives a different insight into the problem from the one that existed before.

In order to concentrate on only the needed information, it is necessary to partition the problem into manageable parts. This is another way of saying that partitioning eliminates much of the needless information gathering and directs the efforts to the essential areas.

Information partitioning is carried out in two ways:

1. Vertically—which involves finding the bottlenecks that limit what can be accomplished and then concentrating on them as if they were the full problem. This usually results in reducing the size of the problem and gets down to the heart or core of the actual problem.

2. Horizontally—which involves partitioning the bottleneck into logical information areas to facilitate the gathering of the data. Generally, these data are put up in tables which simplify combining and manipulating of data. In every case an effort should be made to tabularize the data in a way that makes it convenient to express it in standard or

comparable units. Irrespective of the method used to solve the problem, expressing the data in meaningful units is a basic requirement. If it is possible to express all data in identical or standard units, then one of the simpler methods, such as the modi, can be used.

After you have gained some experience in applying linear programming you may reverse this process on occasion. To be sure, you will need a general idea of the problem as a place to begin. But once you have worked through a few problems you will develop a sense that indicates whether or not a problem is a linear-programming problem or whether additional information is needed.

The second effect you will see as you acquire experience is that you will have a predetermined idea of the method you want to use and you will set up the data collection and analysis around this method.

DETERMINING WHEN THE NECESSARY INFORMATION HAS BEEN OBTAINED

There is no hard-and-fast rule that indicates when all the necessary problem information has been obtained. In the case where the simplex method is used, if all the information is not in the problem the answer, if one can be computed at all, will be meaningless. This is an advantage of the simplex method. However, this does not solve the problem of knowing when all the necessary and important information has been put into the formulation of the problem, because even the simplex can solve the wrong problem. Rather, the answer lies in knowing the problem, developing experience and skill in application, and having knowledge of management functions to draw on. In short, there is no substitute for knowledge and experience. Even with these, however, there are guides and check lists which can be useful.

Check lists make it possible to point to the information needed to solve certain problems. By using the check list as a guide you can be fairly certain of getting most of the information needed.

Most manufacturing linear-programming problems will require some or all of the following general information:

1. The general manufacturing process and sequence of operations performed in producing the products
2. The product line, including all regular and special products
3. The customers by product and their location
4. The rate of production by product, by department, and by machine
5. The customer demand by time period
6. The effective available machine time, including utilization, down time, and the like
7. Detailed cost and price information by product, by operation, and by machine group

8. Freight rates, routing methods, market characteristics, and the like, when marketing and distribution problems are involved

Production planning and control problems will require information and data about the following:

1. Current scheduling practices, such as basic approach, methods, procedures, and controls
2. Range, size, and variety of products and parts
3. Variety, models, and number of machine tools
4. Methods of purchasing, status of delivery promises, and tooling
5. Time standards for running various parts on best and alternate machines
6. Setup times and setup codes
7. Available machine time by group and department, including per cent utilization, down time, delays, and the like
8. Method of assigning people to machines
9. Inventory practices and procedures, including lot quantities

CHAPTER 9

The Model

In the linear-programming sense, a Model is a small-scale version of a larger situation that has all the essential features and characteristics of the larger problem. Solving the model provides an answer and insight into the full-scale problem without getting into the data gathering and analysis of the full-scale problem.

There are different kinds of models. One kind is the *physical model,* which is a large scale *object* reduced considerably in size. The electric train under the Christmas tree is an example of a physical model. It is a small-scale copy of the large, everyday railroad train complete with working parts, lights, whistle, and the like. It performs in the same way as its full-scale counterpart.

Another kind of model is the *mathematical model.* As a model, it too is a small-scale working copy of a full-scale problem. Linear-programming models fall into this class. Mathematical models usually consist of *equations or formulas* developed to relate important features of the problem or situation being studied. By solving the model problem it is possible to tell in advance the kind of results that can be expected before getting into the larger problem. With such advance information, it can be determined whether it is worthwhile to solve the full-scale problem and also what is involved in doing so.

ADVANTAGES OF USING A MODEL

The model is an extremely useful tool to the linear-programming practitioner. Setting up and solving a model should be standard practice before any attempt is made to solve full-scale problems for the following reasons·

1. Savings in time and effort

The answer to the model problem will indicate in advance the desirability of attacking and solving the large-scale problem. In some cases solving the full-scale problem may not be worth the effort required. The

model will indicate this before people and money are tied up in a project that will produce little or no results when completed.

The model also helps to indicate which information is important and which is not. Since it is difficult to determine in advance which information is critical and which will have little or no effect, the model provides a convenient way of establishing the importance of information. Then effort can be concentrated on making the important information as good as possible for solving the major problem.

2. Information is provided for directing effort to most desirable or profitable areas

The model can indicate where the best returns are to be obtained. By changing the values of the variables and solving for different alternatives and types of solutions, the model will indicate the place and problem that will bring the greatest returns when solved.

3. Simplified working procedures can be developed and tested

In some applications the time required to work out solutions has made the answer past history by the time it was calculated. It is important that the final working procedure be *rapid* and *simple* if at all possible. The model, because it can be solved quickly, provides an insight into and a means for checking simplified procedures when they have been worked out.

4. New problems and applications can be explored

Because linear programming is a new technique and because the model can be set up and solved fairly rapidly, many problems and situations that were unsolvable before can now be examined and explored. This is especially true where problems were so large that information was treated piecemeal or so complicated that the relationships were obscure.

USE OF A MODEL FOR PROFIT PLANNING [1]

To demonstrate the usefulness and versatility of the model for advance planning and decision making, let us turn to a specific model.

The model in this case is a profit-planning model of a manufacturing firm. It is to be used to explore and evaluate a number of alternative programs for obtaining maximum gross-profit margin under different conditions. We shall assume that the preliminary survey and collection

of information indicates that the restrictions, alternatives, and other conditions are such that linear-programming methods can be used.

For purposes of the problem let us start with two products,—Product A and Product B. Both products require two machining operations—Operation I and Operation II—to manufacture a complete product for shipment.

The machining operations are carried out on several different machine groups. The first operations for both products *must* be carried out on Machine Group 1, which has 1,000 hours available for the period.

The second operations for both products can be carried out on Machine Group 2, which has 600 hours available; Machine Group 2A (overtime on Machine Group 2), which has 200 hours available; or Machine Group 3, which has 800 hours available. Because there are restrictions on machine capacity and since the second-step operations can be carried out on alternative machines, we can begin to see that this is a possible programming problem.

The rates of production in hours per piece and the profit per piece by combination of machines given to us by the accounting department are listed in Table 9-1.

Table 9-1. Summary of Manufacturing and Accounting Information

Operation	Machine group	Production time, hours per piece						Hours available *
		Product A			Product B			
I	*M*1	.0020	.0020	.0020	.0050	.0050	.0050	Up to 1,000
II	*M*2	.0030			.0080			Up to 600
	*M*2*A* †		.0030			.0080		Up to 200
	*M*3			.0040			.0100	Up to 800
Profit margin per piece (in dollars)		.85	.60	.70	1.60	1.40	1.30	

* Hours available, or machine capacity, has been adjusted to reflect utilization of machines.

† *M*2*A* represents Machine 2 on overtime.

With the information given, the model will show what effective planning information linear programming can provide in the following areas:

1. Determining the gross-manufacturing-margin potential of the present product line

Table 9-2. Determining the Highest Gross Manufacturing Margin

	Profit per piece (in dollars)			.00	.00	.00	.00	.85	.60	.70	1.60	1.40	1.30	
		Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
							Tableau 1							
	.00	P_7	1,000,000	1	0	0	0	2	2	2	5	5	5	
←	.00	P_8	600,000	0	1	0	0	3	0	0	8	0	0	
	.00	P_9	200,000	0	0	1	0	0	3	0	0	8	0	Program 1
	.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
			0	.00	.00	.00	.00	−.85	−.60	−.70	−1.60	−1.40	−1.30	
							Tableau 2							
	.00	P_7	625,000	1	−⅝	0	0	⅛	2	2	0	5	5	
→	1.60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0	
←	.00	P_9	200,000	0	0	1	0	0	3	0	0	8	0	Program 2
	.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
			120,000	.00	.20	.00	.00	−.25	−.60	−.70	0	−1.40	−1.30	
							Tableau 3							
	.00	P_7	500,000	1	−⅝	−⅝	0	⅛	⅛	2	0	0	5	
	1.60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0	
→	1.40	P_5	25,000	0	0	⅛	0	0	⅜	0	0	1	0	Program 3
←	.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
			155,000	.00	.20	.175	.00	−.25	−.075	−.70	0	0	−1.30	

Tableau 4

	.00	P_7	100,000	1	$-5/8$	$-5/8$	$-1/2$	$1/8$	$1/8$	0	0	0	0
←	1.60	P_4	75,000	0	$1/8$	0	0	$3/8$	0	0	1	0	0
	1.40	P_5	25,000	0	0	$1/8$	0	0	$3/8$	0	0	1	0
→	1.30	P_6	80,000	0	0	0	$1/10$	0	0	$2/5$	0	0	1
			259,000	.00	.20	.175	.13	$-.25$	$-.075$	$-.18$	0	0	0

Program 4

Tableau 5

	.00	P_7	75,000	1	$-2/3$	$-5/8$	$-1/2$	0	$1/8$	0	$-1/3$	0	0
→	.85	P_1	200,000	0	$1/3$	0	0	1	0	0	$5/3$	0	0
	1.40	P_5	25,000	0	0	$1/8$	0	0	$3/8$	0	0	1	0
←	1.30	P_6	80,000	0	0	0	$1/10$	0	0	$2/5$	0	0	1
			309,000	.00	.283	.175	.13	0	$-.175$	$-.18$	.667	0	0

Program 5

Tableau 6

	.00	P_7	75,000	1	$-2/3$	$-5/8$	$-1/2$	0	$1/8$	0	$-1/3$	0	0
	.85	P_1	200,000	0	$1/3$	0	0	1	0	0	$5/3$	0	0
←	1.40	P_5	25,000	0	0	$1/8$	0	0	$3/8$	0	0	1	0
→	.70	P_3	200,000	0	0	0	$1/4$	0	0	1	0	0	$5/2$
			345,000	.00	.283	.175	.175	0	$-.075$	0	.667	0	.45

Program 6

Tableau 7

	.00	P_7	66,667	1	$-2/3$	$-2/3$	$-1/2$	0	0	0	$-1/3$	$-1/3$	0
	.85	P_1	200,000	0	$1/3$	0	0	1	0	0	$5/3$	0	0
→	.60	P_2	66,667	0	0	$1/3$	0	0	1	0	0	$8/3$	0
	.70	P_3	200,000	0	0	0	$1/4$	0	0	1	0	0	$5/2$
Highest profit (in dollars)			350,000	.00	.283	.200	.175	0	0	0	.667	.200	.450

Program 7

2. Measuring the value of increasing capacity through overtime
3. Evaluating sales restrictions
4. Anticipating the effect of changes in price
5. Exploring the value of methods improvements

In each case the simplex method will be used to solve the model problem because of the useful by-product information that it provides.

1. Determining the highest gross-manufacturing program of the product line (most profitable product mix at the factory)

The arrangement of problem information for analysis by the simplex method is shown in the Tableau 1, or Program 1, of Table 9-2. The production time—hours per piece—and the hours available have been multiplied by 1,000 to eliminate the decimal numbers and simplify the calculations. The columns headed P_1, P_2, and P_3 represent the different activities or machine combinations for manufacturing Product A. Since there is a different cost for manufacturing Product A on overtime, it is treated as if it were another machine combination. Similarly, columns headed P_4, P_5, and P_6 represent the different machine combinations for producing Product B. Columns P_7, P_8, P_9, and P_{10} represent the idle time on the different machines, and P_0 is the column in which the manufacturing program and profit appear. The row labeled Profit per Piece shows the profits per piece according to the combinations of machines used. Idle time—represented by Columns P_7, P_8, P_9, and P_{10}—shows zero profit, or no profit. The column headed Profit per Piece indicates by the presence of the zeros that the starting point is zero and that Program 1 is made up solely of idle time.

Successive tableaus, or programs, show the step-by-step progress made in reducing the idle time by bringing different amounts of Product A and Product B into solution until the best or highest margin program is obtained in Tableau 7. The best program is indicated when there are no negative values in the base, or bottom row, of a tableau or program. A glance at the Base Row of Program 7 will show that there are no negative values.

Thus the program that provides the greatest margin at the factory is shown in Tableau 7, or Program 7. The interpretation of this program is as follows:

The highest profit margin, read directly from the bottom of the P_0 *Column, is $350,000.*

The greatest profit is obtained when all machine groups are used to manufacture Product A—despite the fact that the profit per piece of Product B is considerably higher.

The manufacturing program for producing a margin of $350,000 is read directly from the P_0 Column of Program 7 and is as follows:

$M1M2$ to produce 200,000 pieces of Product A (P_1)
$M1M2A$ to produce 66,667 pieces of Product A (P_2)
$M1M3$ to produce 200,000 pieces of Product A (P_3)

Total 466,667 pieces of Product A

The amount of time needed on each machine group to achieve the greatest profit margin can be readily established once the number of pieces to be produced has been determined. An over-all schedule can be set up which permits planning to be done and decisions to be made regarding raw material, work-in-process inventory, and number of shifts and employees.

A summary of the information that is obtained by solving the problem in this fashion is listed in Table 9-3.

Table 9-3. Highest Gross Manufacturing Margin
(Most Profitable Program for Manufacturing Product A and Product B)

Operation	Machine group	Product A			Product B			Hours needed	Hours available	W
I	$M1$	400	133.3	400	0	0	0	933.3	1,000	0
II	$M2$	600			0			600	600	283
	$M2A$		200			0		200	200	200
	$M3$			800			0	800	800	175
Number of pieces		200,000	66,667	200,000	0	0	0	2,533.3	2,600	
Profit (in dollars)		170,000	40,000	140,000	0	0	0			

The table shows that all the hours available on Machine Group 1 are *not* used in the most profitable program (933.3 hours of the available 1,000 hours are needed). Frequently this is contrary to the belief of some management personnel who believe that the wheels must be turning every minute to make the highest profit. Running the remaining 66.7 hours will result in an increase in work-in-process inventory from partially completed parts.

The figures which appear in the W column at the right-hand side of the table are an important by-product of the calculations and are obtained from the Base Row of the final program (Program 7). They represent the *gain* in profit that will result from increasing the hours available for production on the specific machine by *1 hour*.

The W value of 283 for Machine Group 2 means that adding 1 hour

Table 9-4. Measuring the Value of Increasing Capacity (1 hour additional overtime)

Profit per piece (in dollars)	Product	P_0	.00 P_7	.00 P_8	.00 P_9	.00 P_{10}	.85 P_1	.60 P_2	.70 P_3	1.60 P_4	1.40 P_5	1.30 P_6	
Tableau 1													
00	P_7	1,000,000	1	0	0	0	2	2	2	5	5	5	
← .00	P_8	600,000	0	1	0	0	3	0	0	8	0	0	
.00	P_9	201,000	0	0	1	0	0	3	0	0	8	0	Program 1
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		0	.00	.00	.00	00	−.85	−.60	−.70	−1.60	−1.40	−1.30	
Tableau 2													
.00	P_7	625,000	1	−⅝	0	0	⅛	2	2	0	5	5	
→ 1 60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0	
← .00	P_9	201,000	0	0	1	0	0	3	0	0	8	0	Program 2
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		120,000	.00	.20	.00	.00	−.25	−.60	−.70	0	−1.40	−1.30	
Tableau 3													
.00	P_7	499,375	1	−⅝	−⅝	0	⅛	⅛	2	0	0	5	
1.60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0	
→ 1 40	P_5	25,125	0	0	⅛	0	0	⅜	0	0	1	0	Program 3
← .00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		155,175	.00	.20	.175	.00	−.25	−.075	−.70	0	0	−1.30	

Tableau 4

	.00	P_7	99 375	1	−5/8	−5/8	−1/2	1/8	1/8	0	0	0	0	
←	1 60	P_4	75,000	0	1/8	0	0	3/8	0	0	1	0	0	
	1.40	P_5	25,125	0	0	1/8	0	0	3/8	0	0	1	0	Program 4
→	1.30	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1	
			259,175	.00	.20	.175	13	− 25	−.075	−.18	0	0	0	

Tableau 5

	.00	P_7	74,375	1	−2/3	−5/8	−1/2	0	1/8	0	−1/3	0	0	
→	.85	P_1	200,000	0	1/3	0	0	1	0	0	5/3	0	0	
	1 40	P_5	25,125	0	0	1/8	0	0	3/8	0	0	1	0	Program 5
←	1 30	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1	
			309,175	.00	283	.175	13	0	− 075	− 18	667	0	0	

Tableau 6

	00	P_7	74,375	1	−2/3	−5/8	−1/2	0	1/8	0	−1/3	0	0	
	85	P_1	200 000	0	1/3	0	0	1	0	0	5/3	0	0	
←	1 40	P_5	25,125	0	0	1/8	0	0	3/8	0	0	1	0	Program 6
→	.70	P_3	200,000	0	0	0	1/4	0	0	1	0	0	5/2	
			345,175	.00	.283	.175	.175	0	− 075	0	.667	0	.45	

Tableau 7

	.00	P_7	66,000	1	−2/3	−2/3	−1/2	0	0	0	−1/3	−1/3	0	
	.85	P_1	200,000	0	1/3	0	0	1	0	0	5/3	0	0	
→	.80	P_2	67,000	0	0	1/3	0	0	1	0	0	8/3	0	Program 7
	.70	P_3	200,000	0	0	0	1/4	0	0	1	0	0	5/2	
Highest profit (in dollars)			350,200	.00	.283	.200	.175	0	0	0	.667	.200	.450	

Table 9-5. Measuring the Value of Increasing Capacity (100 hours additional overtime)

Profit per piece (in dollars)	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
			.00	.00	.00	.00	.85	60	.70	1.60	1.40	1.30	
Tableau 1													
00	P_7	1,000 000	1	0	0	0	2	2	2	5	5	5	Program 1
← 00	P_8	600 000	0	1	0	0	3	0	0	8	0	0	
00	P_9	300,000	0	0	1	0	0	3	0	0	8	0	
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		0	0	0	0	0	− 85	− 60	− 70	−1.60	−1.40	−1 30	
Tableau 2													
.00	P_7	625,000	1	−5/8	0	0	1/8	2	2	0	5	5	Program 2
→ 1 60	P_4	75 000	0	1/8	0	0	3/8	0	0	1	0	0	
← 00	P_9	300 000	0	0	1	0	0	3	0	0	8	0	
00	P_{10}	800 000	0	0	0	1	0	0	4	0	0	10	
		120,000	00	.20	.00	.00	−.25	− 60	−.70	0	−1 40	−1.30	
Tableau 3													
00	P_7	437,500	1	−5/8	−5/8	0	1/8	1/8	2	0	0	5	Program 3
1 60	P_4	75 000	0	1/8	0	0	3/8	0	0	1	0	0	
→ 1 40	P_5	37 500	0	0	1/8	0	0	3/8	0	0	1	0	
← .00	P_{10}	800 000	0	0	0	1	0	0	4	0	0	10	
		172,500	.00	.20	.175	.00	− 25	−.075	− 70	0	0	−1.30	

Tableau 4

	.00	P_7	37,500	1	−5/8	−5/8	−1/2	1/8	1/8	0	0	0	0
←	1.60	P_4	75 000	0	1/8	0	0	3/8	0	0	1	0	0
	1.40	P_5	37 500	0	0	1/8	0	0	3/8	0	0	1	0
→	1.30	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1
			276,500	.00	.20	.175	.13	−.25	−.075	−.18	0	0	0

Program 4

Tableau 5

	00	P_7	12 500	1	−2/3	−5/8	−1/2	0	1/8	0	1/3	0	0
→	.85	P_1	200 000	0	1/2	0	0	1	0	0	8/3	0	0
	1 40	P_5	37 500	0	0	1/2	0	0	3/8	0	0	1	0
←	1 30	P_6	80 000	0	0	0	1/10	0	0	2/5	0	0	1
			326,500	.00	.283	.175	13	0	− 175	− 18	.667	0	0

Program 5

Tableau 6

	.00	P_7	12,500	1	−2/3	−5/8	−1/2	0	1/8	0	−1/3	0	0
	85	P_1	200 000	0	1/3	0	0	1	0	0	8/3	0	0
←	1 40	P_5	37,500	0	0	1/8	0	0	3/8	0	0	1	0
→	.70	P_3	200,000	0	0	0	1/4	0	0	1	0	0	5/2
			362,500	.00	.283	.175	175	0	− 075	0	.667	0	.45

Program 6

Tableau 7

	.00	P_7	0	1	−2/3	−2/3	−1/2	0	0	0	−1/3	−1/3	0
	.85	P_1	200 000	0	1/3	0	0	1	0	0	8/3	0	0
→	.60	P_2	100,000	0	0	1/3	0	0	1	0	0	8/3	0
	.70	P_3	200,000	0	0	0	1/4	0	0	1	0	0	5/2
Highest profit (in dollars)			370,000	.00	.283	.200	.175	0	0	0	.667	.200	.450

Program 7

to the hours available will increase the profit margin $283. The W value for Machine Group 1 is zero, which means that there will be no increase in profits by adding to the hours available. This we know anyway because there is still idle, or unused, time on Machine Group 1 in the best program. There is nothing to be gained by adding hours. Similarly, the W values, or profit margins, shown for Machine Group 2 and 2A have the same significance. The W values remain constant as long as there is unused capacity on Machine Group 1 or until there is perfect balance between Operations I and II.

One of the interesting features of this model is that the W values show a greater profit margin obtained when Machine Group 2 is run on overtime (the W value for $M2A$ is 200) rather than operating Machine Group 3 at straight time ($W = 175$).

The usefulness of the W values can be shown by solving the problem again, first adding *1 hour* to the available capacity of Machine Group 2A and then adding *100 hours* and seeing what the effect is.

2. Measuring the value of increasing capacity through overtime

Adding 1 hour overtime

Adding 1 hour on Machine Group 2A raises the hours available to 201 hours. This change, together with the resulting program and profit, is shown in Table 9-4 on pages 182 and 183.

The highest profit read from the Base Row of Program 7 is $350,200, a $200 increase. This is the additional profit that the W value indicated we would obtain by adding 1 hour to the hours available on Machine Group 2A.

The $200 comes from producing 33⅓ more pieces on Machine 1 and Machine 2A at a profit of $.60 each ($33\frac{1}{3} \times \$.60 = \$200$). The remainder of the program is the same as it was under the highest-gross-manufacturing program.

Adding 100 hours overtime

It is not possible to keep adding overtime without going beyond the point at which it will cease to add to profits. The point at which it no longer becomes profitable can be determined in advance. For example, on Machine Group 2A, it is profitable to continue adding overtime up to and including 100 hours, all other values remaining the same. The 100 hours are determined by referring to Column P_9 in Program 7 of the original product-mix problem. Column 9 represents idle time on Machine Group 2A and in Program 7 has four entries which correspond to four products, P_7, P_1, P_2, and P_3, in the Product Column. Two of the products, P_1 and P_3, have a zero in Column P_9. P_1 represents the amount of Product A to be produced on $M1M2$. P_3 represents the amount of Product A to

be produced on $M1M3$. Neither of these, then, involves $M2A$. We can, therefore, neglect them for this computation.

On the other hand, P_7 (idle time on $M1$) and P_2 (the amount of Product A to be produced on $M1M2A$) have number entries which indicate how much overtime can be added that adds to profit. We know that there are $66\frac{2}{3}$ hours idle time on $M1$. The $-\frac{2}{3}$ value in Column P_9 opposite row entry P_7 (idle time on $M1$) represents the ratio of hours of $M1M2A$ required to produce one unit of Product A. Using the ratio of 2/3, we can add as much as 100 hours on $M2A$ before we use all the idle $66\frac{2}{3}$ hours on $M1$. Beyond this point the W values will change, and additions of overtime will add nothing to profits.

The program and profit that result from addirg 100 hours overtime on $M2A$ is shown in Program 7 of Table 9-5 on page 185.

The highest profit read from the Base Row of Program 7 is $370,000, an increase of $20,000. This confirms that adding 100 hours overtime ($200 + 100 = 300$) at $200 per hour brings an increase in profit of $20,000.

The increase in profits is the result of producing 100,000 pieces of Product A instead of 66,667 pieces on the combination of $M1M2A$ is given in Table 9-6.

Table 9-6. Program of Highest Profit Margin

(100 additional hours overtime)

Product	Machine-group combination						Total to be produced
	$M1M2$	$M1M2A$	$M1M3$	$M1M2$	$M1M2A$	$M1M3$	
A	200,000	100,000	200,000				500,000
B				0	0	0	0
Profit (in dollars)	170,000	60,000	140,000				

Adding 101 hours overtime

In order to prove that 100 hours is the maximum number of overtime hours that can be added to $M2A$ and increase profits, suppose we solve the problem showing 301 hours available on $M2A$. The profit is read from the Base Row in Program 8 of Table 9-7 on page 191. Maximum profit of $370,000 and program for Product A is the same as the previous program.

Table 9-7. Measuring the Value of Increasing Capacity
(101 hours additional overtime)

Profit per piece (in dollars)			.00	.00	.00	.00	.85	.60	.70	1.60	1.40	1.30
	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6

Tableau 1

.00	P_7	1,000,000	1	0	0	0	2	2	2	5	5	5
← .00	P_8	600,000	0	1	0	0	3	0	0	8	0	0
.00	P_9	301,000	0	0	1	0	0	3	0	0	8	0
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10
		0	.00	.00	.00	.00	−.85	−.60	−.70	−1.60	−1.40	−1.30

Program 1

Tableau 2

.00	P_7	625,000	1	−⅝	0	0	⅛	2	2	0	5	5
→ 1.60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0
← .00	P_9	301,000	0	0	1	0	0	3	0	0	8	0
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10
		120,000	.00	.20	00	.00	−.25	−.60	−.70	0	−1.40	−1.30

Program 2

Tableau

	.00	P_7	436,875	1	−5/8	−5/8	0	1/8	1/8	2	0	0	5
	1.60	P_4	75,000	0	1/8	0	0	3/8	0	0	1	0	0
→	1.40	P_5	37,675	0	0	1/8	0	0	3/8	0	0	1	0
←	.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10
			172,675	.00	.20	.175	.00	−.25	−.075	−.70	0	0	−1.30

Program 3

Tableau 4

	.00	P_7	36,875	1	−5/8	−5/8	−1/2	1/8	1/8	0	0	0	0
←	1.60	P_4	75,000	0	1/8	0	0	3/8	0	0	1	0	0
	1.40	P_5	37,625	0	0	1/8	0	0	3/8	0	0	1	0
→	1.30	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1
			276,675	.00	.20	.175	.13	−.25	−.075	− 18	0	0	0

Program 4

Table 9-7. Measuring the Value of Increasing Capacity (*continued*)
(101 hours additional overtime)

Profit per piece (in dollars)			.00	.00	.00	.00	.85	.60	.70	1.60	1.40	1.30	
	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
						Tableau 5							
.00	P_7	11,875	1	−2/3	−5/8	−1/2	0	1/8	0	1/3	0	0	Program 5
→ .85	P_1	200,000	0	1/2	0	0	1	0	0	8/3	0	0	
1.40	P_5	37,625	0	0	1/8	0	0	3/8	0	0	1	0	
← 1.30	P_6	80,000	0	0	0	1/10	0	0	2/3	0	0	1	
		326,675	00	283	.175	.13	0	−.175	−.18	.667	0	0	
						Tableau 6							
← .00	P_7	11,875	1	−2/3	−5/8	−1/2	0	1/8	0	−1/3	0	0	Program 6
.85	P_1	200,000	0	1/3	0	0	1	0	0	8/3	0	0	
1.40	P_5	37,625	0	0	1/8	0	0	3/8	0	0	1	0	
→ .70	P_3	200,000	0	0	0	1/4	0	0	1	0	0	5/2	
		362,675	.00	283	.175	.175	0	−.075	0	.667	0	.45	

Tableau 7

→	.60	P_2	95,000	8	$-16/3$	-5	-4	0	1	0	$-8/3$	0	0	Program 7
	.85	P_1	200,000	0	$1/3$	0	0	1	0	0	$8/3$	0	0	
←	1.40	P_5	2,000	-3	2	2	$3/2$	0	0	0	1	1	0	
	.70	P_3	200,000	0	0	0	$1/4$	0	0	1	0	0	$5/2$	
			369,800	.60	−.117	−.200	−.125	0	0	0	.467	0	.450	

Tableau 8

	.60	P_2	100,000	$1/2$	$-1/3$	0	$-1/4$	0	1	0	$-1/6$	$5/2$	0	Program 8
	.85	P_1	200,000	0	$1/3$	0	0	1	0	0	$5/3$	0	0	
→	.00	P_9	1,000	$-3/2$	1	1	$3/4$	0	0	0	$1/2$	$1/2$	0	
	.70	P_3	200,000	0	0	0	$1/4$	0	0	1	0	0	$5/2$	
Highest profit (in dollars)			370,000	.300	.083	0	.025	0	0	0	.567	.100	.450	

Table 9-8. Evaluating Sales Restrictions (Sales of 100,000 pieces Product B)

Profit per piece (in dollars)			.00	.00	.00	.00	$-M$	.85	.60	.70	1.60	1.40	1.30	
	Product		P_7	P_8	P_9	P_{10}	P_{11}	P_1	P_2	P_3	P_4	P_5	P_6	
Tableau 1														
.00	P_7	1,000,000	1	0	0	0	0	2	2	2	5	5	5	Program 1
← .00	P_8	600,000	0	1	0	0	0	3	0	0	8	0	0	
.00	P_9	200,000	0	0	1	0	0	0	3	0	0	8	0	
.00	P_{10}	800,000	0	0	0	1	0	0	0	4	0	0	10	
$-M$	P_{11}	100,000	0	0	0	0	1	0	0	0	1	1	1	
		$-100,000M$	0	0	0	0	0	$-.85$	$-.60$	$-.70$	$-M -1.60$	$-M -1.40$	$-M -1.30$	
Tableau 2														
.00	P_7	625,000	1	$-5/8$	0	0	0	$1/8$	2	2	0	5	5	Program 2
→ 1.60	P_4	75,000	0	$1/8$	0	0	0	$3/8$	0	0	1	0	0	
← .00	P_9	200,000	0	0	1	0	0	0	3	0	0	8	0	
.00	P_{10}	800,000	0	0	0	1	0	0	0	4	0	0	10	
$-M$	P_{11}	25,000	0	$-1/8$	0	0	1	$-3/8$	0	0	0	1	1	
		$-25,000M$ $+120,000$	0	$\frac{M+1.6}{8}$	0	0	0	$\frac{3M-2}{8}$	$-.60$	$-.70$	0	$-M -1.40$	$-M -1.30$	
Tableau 3														
.00	P_7	500,000	1	$-5/8$	$-5/8$	0	0	$1/8$	$1/8$	2	0	0	5	Program 3
1.60	P_4	75,000	0	$1/8$	0	0	0	$3/8$	0	0	1	0	0	
→ 1.40	P_5	25,000	0	0	$1/8$	0	0	0	$3/8$	0	0	1	0	
.00	P_{10}	800,000	0	0	0	1	0	0	0	4	0	0	10	
← $-M$	P_{11}	0	0	$-1/8$	$-1/8$	0	1	$-3/8$	$-3/8$	0	0	0	1	
		155,000	0	$\frac{M+1.6}{8}$	$\frac{M+1.4}{8}$	0	0	$\frac{3M-2}{8}$	$\frac{15M-3}{40}$	$-.70$	0	0	$-M -1.30$	

Tableau 4

.00	P_7	500,000	1	0	0	0	$-1/5$	2	2	2	0	0	0
← 1.60	P_4	75,000	0	$1/8$	0	0	0	$5/8$	0	0	1	0	0
1.40	P_5	25,000	0	0	$1/8$	0	0	0	$3/8$	0	0	1	0
.00	P_{10}	800,000	0	$5/4$	$5/4$	1	$-1/10$	$15/4$	$15/4$	4	0	0	0
→ 1.30	P_6	0	0	$-1/8$	$-1/8$	0	1	$-3/8$	$-3/8$	0	0	0	1
		155,000	0	3/80	1/80	0	$M + 1.30$	$-59/80$	$-225/400$	$-.70$	0	0	0

Program 4

Tableau 5

.00	P_7	100,000	1	$-2/3$	0	0	$-1/5$	0	2	2	$-16/3$	0	0
→ .85	P_1	200,000	0	$1/3$	0	0	0	1	0	0	$8/3$	0	0
1.40	P_5	25,000	0	0	$1/8$	0	0	0	$3/8$	0	0	1	0
← .00	P_{10}	50,000	0	0	$5/4$	1	$-1/10$	0	$15/4$	4	-10	0	0
1.30	P_6	75,000	0	0	$-1/8$	0	1	0	$-3/8$	0	1	0	1
		302,500	0	17/60	1/80	0	$M + 1.3$	0	$-225/400$	$-.70$	59/30	0	0

Program 5

Tableau 6

.00	P_7	75,000	1	$-2/3$	$-3/8$	$-1/2$	$-3/20$	0	$1/8$	0	$-1/3$	0	0
.85	P_1	200 000	0	$1/3$	0	0	0	1	0	0	$8/3$	0	0
1.40	P_5	25 000	0	0	$1/8$	0	0	0	$3/8$	0	0	1	0
→ .70	P_3	12 500	0	0	$5/16$	$1/4$	$-1/40$	0	$15/16$	1	$-5/2$	0	0
1.30	P_6	75,000	0	0	$-1/8$	0	1	0	$-3/8$	0	1	0	1
Highest profit (in dollars)		311,250	0	17/60	37/160	7/40	$M +$ 1.2825	0	3/32	0	13/60	0	0

Program 6

Measuring the value of adding to the capacity

The most profitable place to which capacity can be added for the product mix and conditions of the problem is Machine Group 2. The W value indicates that a \$283 increase in profit per hour can be obtained up to 100 hours. This information is an indication of the value of purchasing an additional machine for this group. The same type of analysis can be used to evaluate the other machine groups.

3. Evaluating sales restrictions

Sales as well as production restrictions can be included in the calculations and evaluated. For example, if the sales department has sold and promised for delivery a certain amount of Product B, the best or least expensive program which will produce the required amount of B can be worked out and compared to the most profitable program. Such a comparison enables management to evaluate the worth of sales programs and direct sales effort into the most profitable channels from the over-all point of view of the business.

Assume that 100,000 pieces of B have been sold and must be produced and that all other conditions remain the same. The most profitable program under these conditions is shown in Program 6 of Table 9-8 on page 193. The appearance of a $-M$ in the tableau indicates that we *must* produce 100,000 pieces of Product B. The profit for this program is \$311,250. By comparing this to the original profit of \$350,000 it can be seen that by producing 100,000 pieces of Product B \$38,750 in profits are forgone, and the number of pieces of Product A that will be available for delivery is reduced by 254,167 pieces. The program or combination of machines for obtaining this program is indicated in Table 9-9.

Table 9-9 Program of Highest Profit Margin for Sales 100,000 Pieces of Product B

Product	$M1M2$	$M1M2A$	$M1M3$	$M1M2$	$M1M2A$	$M1M3$	Total to be produced
A	200,000	0	12,500				212,500
B				0	25,000	75,000	100,000
Profit (in dollars)	170,000		8,750		35,000	97,500	

The best way to assign the production of Product A and Product B to the machine groups under these conditions is not obvious by pencil-and-paper methods. The programming solution, on the other hand, indicates clearly the program that should be followed. In the same manner, various sales programs can be tested and evaluated from this information. It is sometimes possible to tie salesmen's incentive compensation to product profitability, to encourage a more profitable demand pattern.

4. Anticipating the effect of changes in selling price

The effect of a change in selling price and its effect on profit per piece can be evaluated in terms of the most profitable program incorporating the change. Suppose that in order to remain competitive the selling price of Product A is reduced to a point that the profit per piece for each combination is reduced by $.09. Under these circumstances, the most profit is derived from diverting overtime formerly used to produce Product A to Product B, as indicated in Program 6, where the effect of a $.09 reduction in profit in Product A is measured. This program yields a profit of $309,000. For the first time, Product B appears in the highest-margin program, as shown in Tables 9-10 and 9-11 on pages 196-198. Had the original program been continued (466,667 pieces of Product A, 0 pieces of Product B), the profit would have amounted to $308,000, by use of the lower profit per piece for Product A. In this case, it might be decided to continue with the original program for reasons other than greatest profit since the difference between the two programs is so small. The advantage of going through the process of determining the most profitable program is that it provides the means for determining what the cost is in terms of lost opportunity. For example, further reductions in the selling price of Product A will very soon make it more profitable to manufacture Product B—a point which may not be quickly or accurately determined otherwise.

5. Exploring the value of methods improvements

The worth of methods improvements can be evaluated in terms of additional production and profits. Profit per piece may or may not change with a change in the production time per piece. In either case, the value of a contemplated change can be easily evaluated in relation to expenditures for new equipment, tools, or even research. The true value of a methods change can be determined in this way by measuring its over-all effect on manufacturing facilities. For example, if a contemplated methods change would bring about a reduction in the production time per piece on Machine Group *M*2 and *M*2*A* from .0030 to .0020 hours per

Table 9-10. Anticipating the Effect of Changes in Selling Price

Profit per piece (in dollars)			.00	.00	.00	.00	.76	.51	.61	1.60	1.40	1.30	
	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
Tableau 1													
.00	P_7	1,000,000	1	0	0	0	2	2	2	5	5	5	
← .00	P_8	600,000	0	1	0	0	3	0	0	8	0	0	
.00	P_9	200,000	0	0	1	0	0	3	0	0	8	0	Program 1
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		0	.00	.00	.00	.00	−.76	−.51	−.61	−1.60	−1.40	−1.30	
Tableau 2													
.00	P_7	625,000	1	−5/8	0	0	1/8	2	2	0	5	5	
→ 1.60	P_4	75,000	0	1/8	0	0	3/8	0	0	1	0	0	
← .00	P_9	200,000	0	0	1	0	0	3	0	0	8	0	Program 2
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		120,000	.00	.20	.00	.00	−.16	−.51	−.61	0	−1.40	−1.30	
Tableau 3													
.00	P_7	500,000	1	−5/8	−5/8	0	1/8	1/8	2	0	0	5	
1.60	P_4	75,000	0	1/8	0	0	3/8	0	0	1	0	0	
→ 1.40	P_5	25,000	0	0	1/8	0	0	3/8	0	0	1	0	Program 3
← .00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		155,000	.00	.20	.175	.00	−.16	− 015	−.61	0	0	−1.30	

Tableau 4

	.00	P_7	100,000	1	−5/8	−5/8	−1/2	1/8	1/8	0	0	0	0
←	1.60	P_4	75,000	0	1/8	0	0	3/8	0	0	1	0	0
	1.40	P_5	25,000	0	0	1/8	0	0	3/8	0	0	1	0
→	.00	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1
			259,000	.00	.20	.175	.13	− 16	− 015	−.09	0	0	0

Program 4

Tableau 5

	.00	P_7	75 000	1	−2/3	−5/8	−1/2	0	1/8	0	−1/3	0	0
→	.76	P_1	200,000	0	1/3	0	0	1	0	0	8/3	0	0
	1.40	P_5	25,000	0	0	1/8	0	0	3/8	0	0	1	0
←	1.30	P_6	80 000	0	0	0	1/10	0	0	2/5	0	0	1
			291,000	.00	.253	.175	.13	0	015	− 09	.427	0	0

Program 5

Tableau 6

	.00	P_7	75,000	1	−2/3	−5/8	−1/2	0	1/8	0	−1/3	0	0
	.76	P_1	200,000	0	1/3	0	0	1	0	0	8/3	0	0
	1.40	P_5	25,000	0	0	1/8	0	0	3/8	0	0	1	0
→	.61	P_3	200 000	0	0	0	1/4	0	0	1	0	0	5/2
Highest profit (in dollars)			309,000	0	253	.175	.153	0	.020	0	.427	0	.225

Program 6

Table 9-11. Program of Highest Margin with a Reduction of $.09 Profit in Product A

Amount to produce	Machine-group combination						Totals
	$M1M2$	$M1M2A$	$M1M3$	$M1M2$	$M1M2A$	$M1M3$	
A	200,000	0	200,000				400,000
B				0	25,000	0	25,000
Profit (in dollars)	152,000	0	122,000	0	35,000	0	

piece for Product A, the most profitable program is as shown in Program 9 of Table 9-12.

This program yields a profit of $395,000, which is an increase of $45,000 over the original program. The increase then becomes a yardstick by which the worth of proposed methods improvements can be evaluated. It is significant that it is no longer profitable to utilize overtime on $M2$ ($M1M2A = 0$) under the contemplated program. This is indicated by the final program, shown in Table 9-13 on page 202.

Methods changes (such as new jigs, fixtures, workplace layouts, and conveyors) that will improve the production time per piece for Product B can be worked out in the same manner. In this way it is possible to evaluate methods improvements that cause some of Product B to be included in the most profitable program. For example, introducing methods improvements that change the production time per piece on $M1$ from .0050 to .0040 and on $M3$ from .0100 to .0070 for Product B will cause approximately 114,000 pieces of Product B to be included in the most profitable program, as shown in Program 6 of Table 9-14 on page 204. The resulting profit under these conditions turns out to be $358,571. This means that the cost of the methods improvement cannot exceed $8,571 or the improvement will not pay for itself within the period of time being considered. This is shown in Table 9-15 on page 205.

By using this same approach—that is, by evaluating the worth of an improvement in terms of effects on over-all production and profits—it is possible to show that some methods improvements are not advisable over all, even though they may increase the output for any given machine group. In this way linear programming offers a better way of evaluating a methods improvement program. By using the model, the worth of a

Table 9-12. Exploring the Value of Methods Improvements Affecting Product A

Profit per piece (in dollars)			.00	.00	.00	00	.85	.60	.70	1.60	1.40	1.30	
	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
						Tableau 1							
.00	P_7	1,000,000	1	0	0	0	2	2	2	5	5	5	Program 1
← .00	P_8	600,000	0	1	0	0	2	0	0	8	0	0	
.00	P_9	200,000	0	0	1	0	0	2	0	0	8	0	
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		0	.00	.00	.00	.00	−.85	−.60	−.70	−1.60	−1.40	−1.30	
						Tableau 2							
.00	P_7	625,000	1	−5/8	0	0	3/4	2	2	0	5	5	Program 2
→ 1.60	P_4	75,000	0	1/8	0	0	1/4	0	0	1	0	0	
← .00	P_9	200,000	0	0	1	0	0	2	0	0	8	0	
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		120,000	.00	.20	.00	.00	−.45	−.60	−.70	0	−1.40	−1.30	
						Tableau 3							
.00	P_7	500,000	1	−5/8	−5/8	0	3/4	3/4	2	0	0	5	Program 3
1.60	P_4	75,000	0	1/8	0	0	1/4	0	0	1	0	0	
→ 1.40	P_5	25,000	0	0	1/8	0	0	1/4	0	0	1	0	
← .00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	10	
		155,000	.00	.20	.175	.00	−.45	−.25	−.70	0	0	−1.30	

Table 9-12. Exploring the Value of Methods Improvements Affecting Product A (*continued*)

Profit per piece (in dollars)			.00	.00	.00	00	.85	.60	.70	1.60	1.40	1.30	
	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
Tableau 4													
← .00	P_7	100,000	1	−5/8	−5/8	−1/2	1/4	3/4	0	0	0	0	Program 4
1.60	P_4	75,000	0	1/8	0	0	1/4	0	0	1	0	0	
1.40	P_5	25,000	0	0	1/8	0	0	1/4	0	0	1	0	
→ 1.30	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1	
		259,000	00	.20	.175	13	−.45	− 25	− 18	0	0	0	
Tableau 5													
→ .85	P_1	400,000/3	4/3	−5/6	−5/8	−2/3	1	1	0	0	0	0	Program 5
1.60	P_4	125,000/3	−1/3	1/3	5/24	1/6	0	−1/4	0	1	0	0	
← 1.40	P_5	25,000	0	0	1/8	0	0	1/4	0	0	1	0	
1.30	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1	
		319,000	.60	− 175	− 20	− 17	0	20	−.18	0	0	0	
Tableau 6													
.85	P_1	300,000	4/3	−5/6	0	−2/3	1	8/3	0	0	29/3	0	Program 6
1.60	P_4	0	−1/3	1/3	0	1/6	0	−2/3	0	1	−5/3	0	
.00	P_9	200,000	0	0	1	0	0	2	0	0	8	0	
← 1.30	P_6	80,000	0	0	0	1/10	0	0	2/5	0	0	1	
		359,000	.60	−.175	0	−.170	0	.60	−.18	0	1.60	0	

Tableau 7

	.85	P_1	300,000	4/3	−5/6	0	−2/3	1	8/3	0	0	29/3	0	Program 7
←	1.60	P_4	0	−1/3	1/3	0	1/6	0	−2/3	0	1	−5/3	0	
	.00	P_9	200,000	0	0	1	0	0	2	0	0	8	0	
→	.70	P_3	200,000	0	0	0	1/4	0	0	1	0	0	5/2	
			395,000	.60	−.175	0	−.125	0	.60	0	0	1.60	450	

Tableau 8

	.85	P_1	300,000	1/2	0	0	−1/4	1	1	0	5/2	5/2	0	Program 8
⇄	.00	P_8	0	−1	1	0	1/2	0	−2	0	3	−5	0	
	.00	P_9	200,000	0	0	1	0	0	2	0	0	8	0	
←	.70	P_3	200,000	0	0	0	1/4	0	0	1	0	0	5/2	
			395,000	.425	0	0	−.0375	0	.250	0	5.25	.725	.450	

Tableau 9

	.85	P_1	300,000	0	1/2	0	0	1	0	0	4	0	0	Program 9
	.00	P_{10}	0	−2	2	0	1	0	−4	0	6	−10	0	
	.00	P_9	200,000	0	0	1	0	0	2	0	0	8	0	
→	.70	P_3	200,000	1/2	−1/2	0	0	0	1	1	−3/2	5/2	5/2	
Highest profit (in dollars)			395,000	.350	.0750	0	0	0	.100	0	.750	.350	.450	

Table 9-13. Program of Highest Profit Margin Methods Improvements Affecting Product A

Product	Machine-group combination						Amount to produce
	M1M2	*M1M2A*	*M1M3*	*M1M2*	*M1M2A*	*M1M3*	
A	300,000	0	200,000				500,000
B				0	0	0	0
Profit (in dollars)	255,000	0	140,000	0	0	0	

contemplated change can be evaluated in terms of over-all effect and not in relation to one product or one machine group. In addition to measuring the value of contemplated improvements, these methods indicate and point out to management areas in which it might be worthwhile to explore the possibilities of improvement.

The illustrations in this section have been made practical by the use of a small model rather than a full-scale problem. A full-scale problem would, by its very magnitude, hopelessly complicate demonstrations of this sort and render them useless as textbook material. Small models are essential tools of linear-programming training. They are also useful in explaining to management how the results in a full-scale solution were obtained.

In the same way, small-model problems can be used in practical applications to accomplish the results discussed earlier in this section.

1. Time and effort are saved when the applicability of the method and the importance of the data are being determined.

2. Information is provided for directing the effort to where the greatest potential for improvement lies.

3. Simplified working procedures can be developed and tested.

4. New problems and applications can be explored with minimum effort.

LP AS A MANAGEMENT-INTEGRATING PROCEDURE

The several variations of this problem illustrate the use of linear programming for management integration. This use may not be readily apparent for several reasons. First, many people do not see the need for integrating a company program. Second, others who do see the need do

Table 9-14. Exploring the Value of Methods Improvements Affecting Product B

Profit per piece (in dollars)			.00	.00	.00	.00	.85	.60	.70	1.60	1.40	1.30	
	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
Tableau 1													
.00	P_7	1,000,000	1	0	0	0	2	2	2	4	4	4	
← .00	P_8	600,000	0	1	0	0	3	0	0	8	0	0	
.00	P_9	200,000	0	0	1	0	0	3	0	0	8	0	Program 1
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	7	
		0	.00	.00	.00	.00	−.85	−.60	−.70	−1.60	−1.40	−1.30	
Tableau 2													
.00	P_7	700,000	1	−½	0	0	½	2	2	0	4	4	
→ 1.60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0	
← .00	P_9	200,000	0	0	1	0	0	3	0	0	8	0	Program 2
.00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	7	
		120,000	.00	.20	.00	.00	−.25	−.60	−.70	0	−1.40	−1.30	
Tableau 3													
.00	P_7	600,000	1	−½	−½	0	½	½	2	0	0	4	
1.60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0	
→ 1.40	P_5	25,000	0	0	⅛	0	0	⅜	0	0	1	0	Program 3
← .00	P_{10}	800,000	0	0	0	1	0	0	4	0	0	7	
		155,000	.00	.200	.175	.00	−.25	−.075	−.70	0	0	−1.30	

Table 9-14. Exploring the Value of Methods Improvements Affecting Product B (*continued*)

Profit per piece (in dollars)	Product	P_0	P_7	P_8	P_9	P_{10}	P_1	P_2	P_3	P_4	P_5	P_6	
			.00	.00	.00	.00	.85	.60	.70	1.60	1.40	1.30	
Tableau 4													
.00	P_7	1,000,000/7	1	−½	−½	−4/7	½	½	−2/7	0	0	0	Program 4
← 1.60	P_4	75,000	0	⅛	0	0	⅜	0	0	1	0	0	
1.40	P_5	25,000	0	0	⅛	0	0	⅜	0	0	1	0	
→ 1.30	P_6	800,000/7	0	0	0	1/7	0	0	4/7	0	0	1	
		2,125,000/7	.00	.200	.175	.186	−.25	−.075	.043	0	0	0	
Tableau 5													
00	P_7	300,000/7	1	−½	−½	−4/7	0	½	−2/7	−4/3	0	0	Program 5
→ .85	P_1	200,000	0	⅓	0	0	1	0	0	8/3	0	0	
← 1.40	P_5	25,000	0	0	⅛	0	0	⅜	0	0	1	0	
1.30	P_6	800,000/7	0	0	0	1/7	0	0	4/7	0	0	1	
		2,475,000/7	.00	.283	.175	.186	0	−.075	.043	.667	0	0	
Tableau 6													
.00	P_7	200,000/21	1	0	0	−4/7	0	0	−2/7	−4/3	0	0	Program 6
.85	P_1	200,000	0	⅓	0	0	1	0	0	8/3	0	0	
→ .60	P_2	200,000/3	0	0	⅓	0	0	1	0	0	8/3	0	
1.30	P_6	800,000/7	0	0	0	1/7	0	0	4/7	0	0	1	
Highest profit (in dollars)		358,571.43	.00	.283	.200	.186	0	0	.043	.667	.200	0	

not expect to find a way of satisfying the need in a mathematical technique.

The examples that have been solved provide a way of measuring the benefits from the following courses of action:

1. Scheduling more effectively
2. Reducing cost to increase margin of profit
3. Improving methods to increase capacity
4. Anticipating price changes
5. Planning sales incentives
6. Subcontracting
7. Other programs

Each of these programs can result in added profits. The successful execution of each of these programs will change the competitive position of a company. At the same time, these programs can only be executed at the expense of not executing some other program. All these facts and conditions can be organized into an over-all profit picture for the firm. Management can then select a program that is best suited to the over-all objectives and financial budget.

Linear programming will provide information for making the best use of the company's capital, sales, manufacturing, and engineering capacity and ability. This use of linear programming can be far more important than developing the best schedule for a group of machines. The over-all program, however, cannot be put together until its parts are clearly delineated. In this way linear programming is expanding the field of scientific management. It provides further progress in the use of the facts in arriving at an integrated program for a business.

Table 9-15. Program of Highest Profit Margin
Methods Improvements Affecting Product B

Product	Machine-group combination						Amount to produce
	*M*1*M*2	*M*1*M*2*A*	*M*1*M*3	*M*1*M*2	*M*1*M*2*A*	*M*1*M*3	
A	200,000	66,667	0				266,667
B				0	0	114,285	114,285
Profit (in dollars)	170,000	40,000	0	148,571	0	0	

CHAPTER 10

Maximizing Profit Margin Considering Manufacturing and Distribution Costs

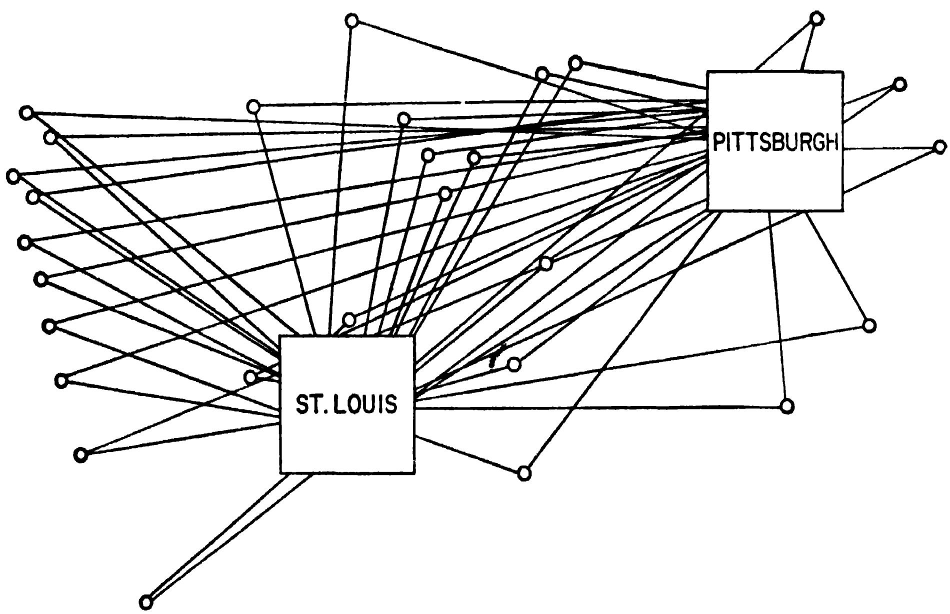


FIG. 10-1. The Walcab Company—Plants and Markets

One of the planning areas in which linear-programming information has proved valuable to management has been in deciding where to assign customers' orders when they can be produced in several plants.

Quite frequently, when plants are in different locations and manufacturing costs vary from plant to plant, there is an opportunity for management to use the information provided by linear programming to increase profits and at the same time make the planning task easier. This was demonstrated in an actual case, here represented by the Walcab Company. Executive management wanted information and a program to enable them to make decisions about the following problem:

In which plant should the various products be manufactured in order to meet customer delivery promises with greatest profit margin to the company considering freight, as well as manufacturing costs?

Some of the conditions which the production-planning people had to consider in developing the answer to this problem were as follows:

1. The different production lines produce at different rates of production and have different costs. Although the total production capacity is almost always more than adequate to meet the demand, the variations from plant to plant and line to line complicate the problem.

2. A number of customers buy on a freight-equalized basis. This method of pricing makes it necessary for the manufacturer to pay part of the freight cost in order to be competitive in certain market areas. This expense becomes a factor when deciding from which plant a given item is to be shipped.

3. One customer purchases items f.o.b. at the manufacturer's plant with the understanding that the supplier will try to ship to each of the customer's destinations from the nearest manufacturing plant.

The headquarters planning group decided to compare an actual record of a previous period with the program for the same period using the LP technique. The comparison provided them with information as follows:

1. The basic maximum-profit program calculated by LP indicated a potential profit increase of $110,000 for the period. This represented a 6 per cent increase in profits obtainable from existing facilities and manpower.

2. Although the policy of shipping f.o.b. at the nearest plant was not being closely followed, it was costing $40,000 in profit to save the customer $19,000 in freight. If the minimum-freight policy had been carried out to the maximum extent, the manufacturer would have been forgoing $218,000 in profits for the year to save the customer $30,000 in freight.

3. A variation of the maximum-profit program, in which one of the most efficient production lines was shut down, showed that transferring orders to other lines could be made to hold the loss in profit to $11,000. This was interesting for two reasons: (1) the loss was not as large as was expected, and (2) the changes in allocations that resulted in the best program under the new conditions were not the ones that management would have chosen as being the easy way out.

The Walcab Company problem which follows demonstrates how planning personnel can use linear-programming methods to develop information to guide their decisions in problems of this kind. The first part of the material describes the operation and gives the basic information needed to solve the problem. Tables of information are shown at this point to indicate the *kind* of information needed.

The second part of the material presents the solution obtained using the modi method described in Section II, "Methods." For ease of reading, the tables of converted information used to solve the problem have been placed at the end of this chapter.

BACKGROUND INFORMATION AND DESCRIPTION OF THE COMPANY OPERATION

The Walcab Company manufactures a complete line of metal wall cabinets for use in home kitchens. There are two plants that supply customers throughout the country. One plant is located in Pittsburgh and the other in St. Louis, both near steel mills that supply the steel sheets used in the product lines.

The Walcab Company does not have its own dealers or retail outlets. Instead it manufactures wall cabinets for sale and distribution to department stores, national mail-order chains, plumbing supply houses, and large private-home construction companies. Customers are located in principal cities throughout the United States. All of them have more than one location requiring cabinets throughout the year.

In order to be competitive in different markets the company absorbs the cost of freight into that market area when necessary. This cost is termed *freight equalization.*

The general manufacturing process and sequence of operations performed in fabricating the product line in Pittsburgh and St. Louis is shown in Figure 10-2.

A study has indicated that the Spray-painting Department in each plant is the bottleneck department and determines the number of cabinets that are fabricated and shipped from each plant. As a result, the Spray-painting Department works three shifts per day, five days a week, and the other departments work one full shift with a skeleton force in Assembly and Inspection on the second shift.

The Spray-painting Department in the Pittsburgh plant has two paint booths and bake ovens. The St. Louis plant has three paint booths and bake ovens. The St. Louis plant is the older and less efficient plant, while the Pittsburgh plant, acquired since 1950, in general has a higher rate of production for a given cabinet size.

The paint booths in both plants are fed by an overhead continuous straight-line conveyor, which moves the parts in front of the painter, who paints them as they move through the paint booth. After they are painted, all parts are carried by the conveyor into an oven adjoining the paint booth, where the painted parts are baked for a hard finish. From the bake ovens the conveyor winds its way into the Shipping and Packing Department before working its way back to the loading point in front of the spray booths. In the Shipping and Packing Department, unloaders

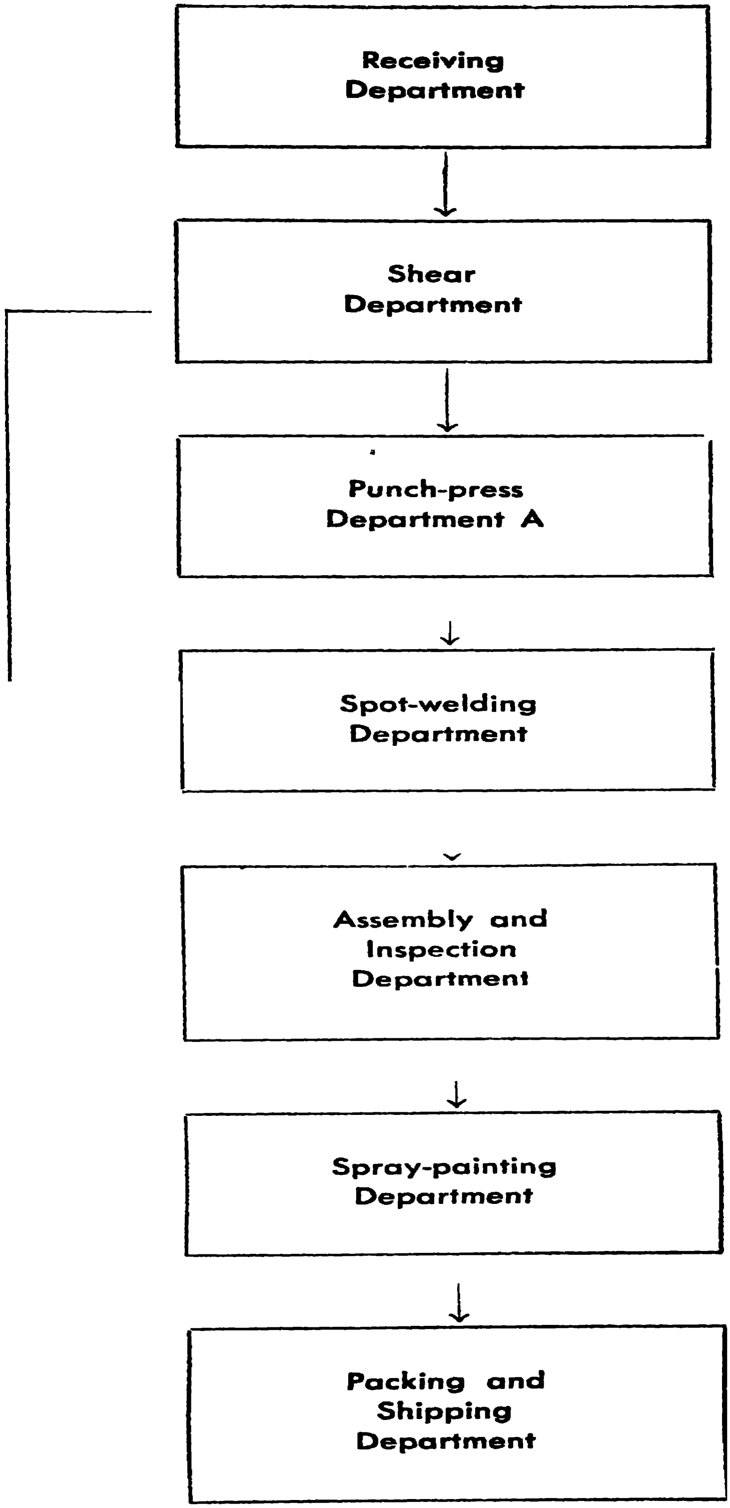


FIG. 10-2. General Manufacturing Process Chart
Pittsburgh and St. Louis Plants

remove the parts from the conveyor and place them in packing cartons, which are sealed, labeled, and sent either to boxcars for immediate shipment or to stock for future shipments.

Spray-paint lines and paint booths are cleaned over the weekend so that the first shift is ready to start on Monday. The bake ovens, which are gas-fired, are turned down over the weekend but started up again in sufficient time to stabilize the furnace temperature before the first shift begins on Monday morning.

The cost of processing any part is different in different booths because of the chain speeds and the spacing of parts permitted by the structure of the chain mechanism.

Product line

Cabinets are made in eight standard sizes and are the type that can be hung easily from a wall bracket. They come in two standard heights, 30 inches and 18 inches. All cabinets are 13 inches deep. Figure 10-3 gives information about cabinet sizes.

Standard Wall-cabinet Sizes

Length (front) (in inches)	Height (in inches)
15	30
18	30
24	30
30	30
36	30
24	18
30	18
36	18

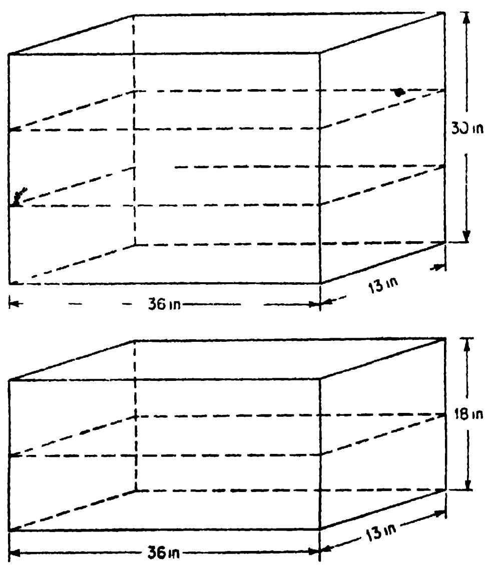


Fig. 10-3. Wall-cabinet Sizes

Cabinets are identified by the letter *W*, meaning "wall," and then in order by the length (front) dimension in inches followed by the height dimension in inches. An example of identification is as follows:

W 3630

This represents a wall cabinet 36 inches long and 30 inches high. The depth dimension is omitted because it is the same—13 inches—for all cabinets.

Customers

Customers and their stores and outlets are located in principal cities throughout the United States. Customers may order any of the standard sizes for any of their outlets. In any quarter, it usually happens that each outlet takes only a few sizes and not the complete range of sizes available. In some cases a given store may not receive anything in a given period.

A list of customers by geographic location is given in Table 10-1.

Table 10-1. Customer List by Geographic Location

Customer	Location	Customer	Location
Central Stores	Seattle	Apex Land	New Orleans
(CS)	Los Angeles	(AL)	Mobile
	Houston		Birmingham
	Atlanta		St. Petersburg
	Cleveland	Western Mail	Denver
	Philadelphia	(WM)	Los Angeles
	Boston		Las Vegas
Ward Gomery	San Francisco		Salem
(WG)	Seattle		Portland
	Dallas		San Francisco
	Kansas City	Eastern Mail	Bridgeport
	Atlanta	(EM)	Trenton
	Detroit		New York
	Pittsburgh		Philadelphia
	New Orleans		Baltimore
	Miami		Wilmington
Jensen Company	Minneapolis		Charleston
(JC)	Chicago	Stacey Construction	Philadelphia
	Detroit	(SC)	
	Cleveland		
	Fort Wayne	Morristown Supply	Morristown
	Indianapolis	(MS)	

Manufacturing process

Since the paint departments are the bottleneck departments, they limit the amount of production and profit that can be obtained. They represent, then, the focal point at which the linear-programming techniques are applied. When those departments have been programmed, scheduling and loading can be worked out for the departments that precede and follow.

The paint departments consist of the spray booths, bake ovens, and

loading areas. The parts to be sprayed are hung by wire hooks to conveyor hooks attached to the conveyor chain. The hanging pattern, or the way in which parts are hung from the conveyor, is important to the amount of production obtained. Hanging patterns have been worked out and methods established by the Industrial Engineering Department.

The Pittsburgh plant has two conveyors, labeled Line A and Line B. The St. Louis plant has three conveyors, labeled Line C, Line D, and Line E.

Hanging patterns

Shells, doors, and shelves are hung on conveyors by wire hooks, which attach to the conveyor hooks and holes in the part being hung.

There must be at least 6 inches between successive parts on the conveyor for the parts to go around turns in the bake oven without bumping each other and marring the paint. A 6-inch clearance also enables the painters to paint the edges and bottoms of parts.

The height of the conveyor is such that the painters have difficulty painting parts that hang lower than 38 inches from the conveyor hooks. Hanging patterns, therefore, cannot exceed 38 inches, including the 4-inch hooks used to hang parts to the conveyor chain.

Painted parts cannot be stocked without wrapping and careful handling because the paint finish damages easily. As a result, there is little stocking of painted parts.

There is a shortage of storage space also. Therefore, cabinets are hung as complete units so that they can be assembled and packed into cartons as they come off the conveyor line. A complete unit consists of the cabinet shell and the required number of doors and shelves. If a group of W 3030 cabinets were being run, for example, first would come the 30- by 13- by 30-inch shell, followed by two 15- by 30-inch doors, and two 13- by 30-inch shelves. The same pattern would repeat again and again until the order was completed. Minor modifications are made to the hanging pattern in order to combine certain parts for more efficient painting and assembly patterns. The rule is, however, that *complete units are run.* It has proved impractical and costly to paint all the shells one after the other, followed later by the doors and shelves for those shells.

It is not possible when loading the conveyor to hang the hooks for two adjacent parts from the same conveyor hook. Some of the typical hanging patterns are as shown in Figure 10-4.

Conveyor hooks per completed cabinet

The distance between centers of consecutive conveyor hooks, the cabinet size, and the number of parts per cabinet determine the number

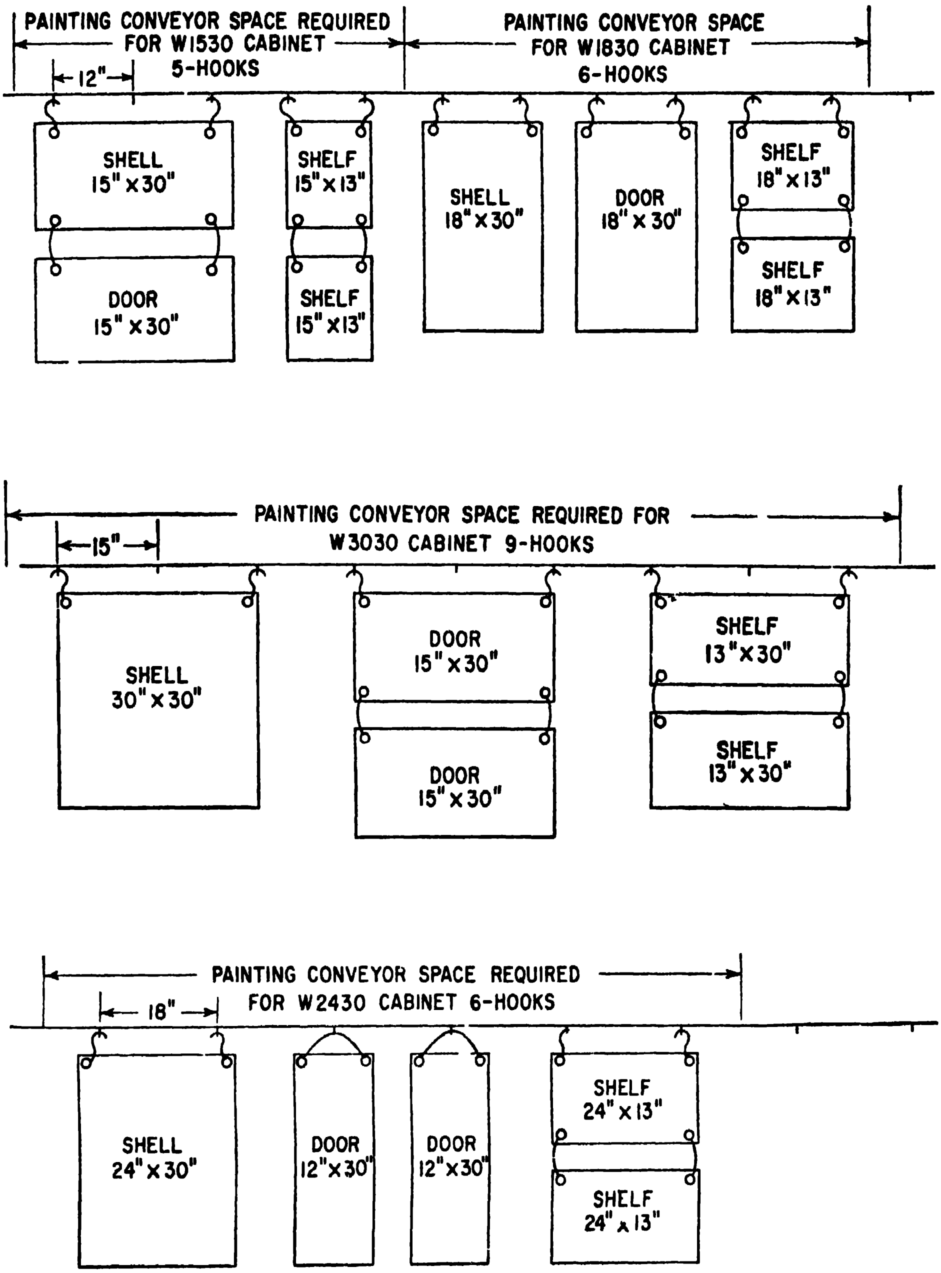
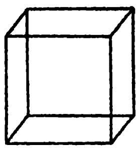
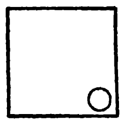

PAINTING CONVEYOR SPACE REQUIRED FOR W1530 CABINET
5-HOOKS
12"
SHELL 15" x 30"
DOOR 15" x 30"
SHELF 15" x 13"
SHELF 15" x 13"
PAINTING CONVEYOR SPACE FOR W1830 CABINET
6-HOOKS
SHELL 18" x 30"
DOOR 18" x 30"
SHELF 18" x 13"
SHELF 18" x 13"
PAINTING CONVEYOR SPACE REQUIRED FOR W3030 CABINET 9-HOOKS
15"
SHELL 30" x 30"
DOOR 15" x 30"
DOOR 15" x 30"
SHELF 13" x 30"
SHELF 13" x 30"
PAINTING CONVEYOR SPACE REQUIRED FOR W2430 CABINET 6-HOOKS
18"
SHELL 24" x 30"
DOOR 12" x 30"
DOOR 12" x 30"
SHELF 24" x 13"
SHELF 24" x 13"

Fig. 10-4.

of hooks that each cabinet requires for a complete unit. The number and size of parts by cabinet are listed in Table 10-2. The number and

Table 10-2. Parts Breakdown by Cabinet Size as Hung on Conveyor for Spray Painting

Cabinet size	Parts					
	Shell 		Door 		Shelf 	
	No.	Size (in inches)	No.	Size (in inches)	No.	Size (in inches)
W 1530	1	15 × 13 × 30	1	15 × 30	2	15 × 13
W 1830	1	18 × 13 × 30	1	18 × 30	2	18 × 13
W 2430	1	24 × 13 × 30	2	12 × 30	2	24 × 13
W 3030	1	30 × 13 × 30	2	15 × 30	2	30 × 13
W 3630	1	36 × 13 × 30	2	18 × 30	2	36 × 13
W 2418	1	24 × 13 × 18	2	12 × 18	1	24 × 13
W 3018	1	30 × 13 × 18	2	15 × 18	1	30 × 13
W 3618	1	36 × 13 × 18	2	18 × 18	1	36 × 13

size of the parts and the hanging patterns according to the distance between conveyor hooks determine the number of conveyor hooks for a complete cabinet. A summary of the number of hooks by cabinet size is given in Table 10-3.

Rate of production

One of the factors upon which the rate of production of a complete cabinet depends is the chain, or conveyor, speed.

At Pittsburgh Line A moves at a rate of 120 feet per hour, and the hooks are placed at 12-inch centers. This means that 120 hooks pass by a given point in the paint booth each hour. Line B moves at a rate of 80 feet per hour, with hooks at 12-inch centers, so that 80 hooks per hour pass each painter on conveyor Line B.

At the St. Louis Plant, Line C and Line D each move at a rate of 60 feet per hour. Since they have hooks at 12-inch centers, 60 hooks per

Table 10-3. Number Conveyor Hooks Required per Complete Cabinet Unit

Cabinet size	Hooks placed at 12-in. centers	Hooks placed at 15-in. centers
W 1530	5	5
W 1830	6	6
W 2430	9	6
W 3030	9	9
W 3630	12	10
W 2418	$5\frac{1}{2}$	5
W 3018	6	$5\frac{1}{2}$
W 3618	8	$7\frac{1}{2}$

hour move into each paint booth. Line E moves at a rate of 120 feet per hour. The hooks are at 15-inch centers, so that 96 hooks pass each painter each hour.

Knowing the hooks per hour and the number of hooks per cabinet, we can compute the number of cabinets per hour that can be painted on each conveyor line. This information is shown in Table 10-4.

Table 10-4. Production Rate per Hour of Completed Cabinets by Conveyor Line

Cabinet size	Pittsburgh		St Louis		
	Line A—12 in. 120 ft per hr 120 hooks per hr	Line B—12 in. 80 ft per hr 80 hooks per hr	Line C—12 in. 60 ft per hr 60 hooks per hr	Line D—12 in. 60 ft per hr 60 hooks per hr	Line E—15 in. 120 ft per hr 96 hooks per hr
W 1530	24	16	12	12	$19\frac{1}{5}$
W 1830	20	$13\frac{1}{3}$	10	10	16
W 2430	$13\frac{1}{3}$	$8\frac{8}{9}$	$6\frac{2}{3}$	$6\frac{2}{3}$	16
W 3030	$13\frac{1}{3}$	$8\frac{8}{9}$	$6\frac{2}{3}$	$6\frac{2}{3}$	$10\frac{2}{3}$
W 3630	10	$6\frac{2}{3}$	5	5	$9\frac{3}{5}$
W 2418	$21\frac{9}{11}$	$14\frac{6}{11}$	$10\frac{10}{11}$	$10\frac{10}{11}$	$19\frac{1}{5}$
W 3018	20	$13\frac{1}{3}$	10	10	$17\frac{5}{11}$
W 3618	15	10	$7\frac{1}{2}$	$7\frac{1}{2}$	$12\frac{4}{5}$

***Table* 10-5. Orders Received by Region for Release and Shipment in the Second Quarter**

Orders received up to and including January

Western region

CS	Los Angeles	2,000—W 3030
CS	Seattle	500—W 3018
CS	Houston	1,200—W 3030
CS	Houston	600—W 1530
WG	San Francisco	100—W 3030
WG	San Francisco	100—W 1530
WG	Dallas	500—W 3630
WG	Dallas	500—W 1830

Middle Western region

CS	St. Louis	2,500—W 3030
JC	Minneapolis	50—W 3630
CS	Cleveland	500—W 3630
CS	Cleveland	500—W 1830

Southern region

AL	New Orleans	250—W 3030
AL	New Orleans	250—W 3018
CS	Atlanta	500—W 3030
CS	Atlanta	1,500—W 3018
AL	St. Petersburg	300—W 1530
AL	St. Petersburg	150—W 3030

Eastern region

EM	Baltimore	250—W 3030
EM	Baltimore	120—W 2418
CS	Philadelphia	100—W 3630
CS	Philadelphia	150—W 3018
MS	Morristown	4,000—W 3630
MS	Morristown	2,500—W 3018

Orders received during February

Western region

WM	Denver	240—W 3030
WM	Los Angeles	620—W 3030
WM	Los Angeles	100—W 3630
WM	Los Angeles	920—W 3018

Middle Western region

WG	Kansas City	1,400—W 3018
WG	Detroit	500—W 3630
JC	Minneapolis	50—W 3630
JC	Minneapolis	40—W 1530
JC	Detroit	1,000—W 3030
JC	Detroit	500—W 1530

Southern region

WG	Atlanta	600—W 3630
WG	Atlanta	200—W 3018
AL	Mobile	150—W 3030
AL	Mobile	300—W 1530

Eastern region

CS	Boston	1,200—W 3030
WG	Pittsburgh	50—W 3630
WG	Pittsburgh	50—W 1530
EM	Wilmington	3,600—W 3630
SC	Philadelphia	9,600—W 1530
SC	Philadelphia	2,880—W 2418
EM	Baltimore	600—W 2418

Orders received during March

Western region

CS	Los Angeles	1,000—W 3630
WM	Denver	360—W 3030
WM	Denver	2,200—W 1830
WM	Denver	1,000—W 3630
WM	Salem	500—W 3630
WM	Salem	240—W 2418
WM	Portland	1,100—W 3030

Middle Western region

JC	Minneapolis	100—W 3630
JC	Minneapolis	200—W 1530
JC	Detroit	100—W 1530
JC	Chicago	500—W 3030
JC	Chicago	525—W 1830
JC	Fort Wayne	800—W 3630

Southern region

WG	New Orleans	100—W 3630
WG	New Orleans	650—W 1530
AL	Birmingham	120—W 1530
AL	Birmingham	110—W 3030
AL	Birmingham	330—W 3630

Eastern region

WG	Pittsburgh	200—W 3030
EM	Bridgeport	120—W 2418
EM	Bridgeport	180—W 1530
EM	Bridgeport	170—W 3030
EM	Trenton	2,000—W 3630
EM	Trenton	2,000—W 3018

Customer demand

The list of orders received by the Sales Department by month for release and shipment is given in Table 10-5.

Available conveyor time

The available conveyor time for the second quarter needs to be determined before the customer demand can be programmed and allocated to each plant.

The Industrial Engineering Department again provides much of the data and figures for computing the available time. One item of information important to the computation is the per cent operative time. This figure is an average percentage used to adjust the maximum available time for missed hooks, reruns of rejects or poor-quality items, unavoidable interruptions, and the like. The per cent operative time reflects the utilization obtained from each conveyor line.

The Maximum Available Time is the product of the number of days per week times the number of shifts per day times the number of hours per shift times the number of weeks per quarter. For example, the maximum number of available hours per quarter is the product of 8 hours per shift $\times$ 3 shifts per day $\times$ 5 days per week $\times$ 13 weeks per quarter = 1,560 hours.

The available time for programming purposes is the product of the maximum available and the per cent operative time. The available conveyor time by conveyor is shown in Table 10-6.

Table 10-6. Available Conveyor Time

Conveyor line	Maximum available time per quarter (in hours)	Per cent operative time	Available time (in hours)
Pittsburgh			
A	1,560	96.0	1,497.60
B	1,560	87.5	1,365.00
St. Louis			
C	1,560	85.5	1,333.80
D	1,560	95.0	1,482 00
E	1,560	89.3	1,393.08
Total			7,071.48

Selling-price information

Information is needed from the Accounting Department and the Billing Department since costs or profits are to be considered. The Billing Department information is shown in Table 10-7, in which the cabinet selling price is the price actually billed.

Table 10-7. Cabinet Selling Price
(In dollars)

Cabinet size	W 1530	W 1830	W 2430	W 3030	W 3630	W 2418	W 3018	W 3618
Selling price	6.50	7.70	10.80	11.27	13.83	7.38	8.00	9.78
Stacey Construction	6.00	7.00	9.80	10.10	12.50	6.75	7.20	8.90
Morris Supply	6.00	7.00	9.80	10.10	12.50	6.75	7.20	8.90

Freight information

Market conditions are such that to be competitive in particular market areas the company has to absorb certain freight charges, which amount to an additional cost. The additional costs per cabinet involved in shipping from each of the manufacturing plants to each of the market areas are given in Table 10-8.

Cost information

Other necessary accounting information is shown in Tables 10-9, 10-10, and 10-11 on pages 221 and 222. Standard costs are used. Fixed salaries and selling expenses are not included.

ACTUAL ASSIGNMENT OF CUSTOMERS' ORDERS FOR THE SECOND QUARTER

The normal practice used to assign customers' orders to the two plants is to consider the freight costs to the destination first. In general, the procedure is to fill each customer order from the plant nearest to the destination wanted. For example, orders for Los Angeles would be filled from the St. Louis plant, while Philadelphia orders would be filled from Pittsburgh.

The actual assignment of customers' orders that was followed in the second quarter is shown in Table 10-12 on page 223. The 63 production

Table 10-8. Freight Equalization Rate per Cabinet—Pittsburgh and St. Louis
(In dollars)

Cabinet size	W 1530		W 1830		W 2430		W 3030		W 3630		W 2418		W 3018		W 3618	
To / From	Pitts.	St. L.	Pitts.	St. L.	Pitts	St. L.	Pitts.	St. L.	Pitts.	St. L.	Pitts.	St. L.	Pitts.	St. L.	Pitts.	St. L.
Los Angeles	.41	.36	.48	.42	.63	.55	.74	.65	.88	.77	.46	.40	.48	.42	.62	.54
Seattle	.41	.36	.48	.42	.63	55	.74	.65	.88	.77	.46	.40	.48	.42	.62	.54
San Francisco	.41	.36	.48	.42	.63	.55	74	.65	.88	.77	.46	.40	.48	.42	.62	.54
Houston	.36	.26	.42	.30	.55	.39	.77	.55	.77	.55	.40	.28	.42	.30	.54	.38
Dallas	.36	.26	.42	.30	.55	.39	.65	.47	.77	.55	.40	.28	.42	.30	.54	.38
Denver	36	.26	.42	.30	.55	.39	.65	.47	.77	55	.40	.28	.42	.30	.54	.38
Salem	.41	.36	.41	.42	.63	.55	.74	.65	.88	.77	.46	.40	.48	.42	.62	.54
Portland	.41	.36	.48	.42	.63	.55	.74	.65	.88	.77	.46	.40	.48	.42	.62	.54
Indianapolis	.61	.16	.18	.18	.23	.20	.29	.29	.33	.33	.16	.16	.18	.18	.22	.22
St. Louis	.21	0	.24	0	.31	0	.38	0	.44	0	.22	0	.24	0	.30	0
Minneapolis	.26	.21	.30	.24	.39	.31	.47	.38	.55	.44	.28	.22	.30	.24	.38	.30
Cleveland	.16	.21	.22	.34	.23	.31	.29	.38	.41	.56	.16	.22	.18	.24	.22	.30
Kansas City	.16	.21	.18	24	.23	.31	.29	.38	.33	.44	.16	.22	.18	.24	.22	.30
Detroit	.18	.25	.18	.24	.23	.31	.36	47	.38	.53	.16	.22	.18	.24	.22	.30
Chicago	.16	.16	.20	.21	.23	.23	.36	.38	.33	.33	.16	.16	.18	.18	.22	.22
Fort Wayne	.16	.16	.18	.18	.23	.23	.29	.29	.33	33	.16	.16	.18	.18	.22	.22

Table 10-8. Freight Equalization Rate per Cabinet–Pittsburgh and St. Louis (*continued*)
(In dollars)

Cabinet size	W 1530		W 1830		W 2430		W 3030		W 3630		W 2418		W 3018		W 3618	
To / From	Pitts.	St. L.	Pitts	St. L.	Pitts.	St. L.	Pitts	St. L.	Pitts	St. L.	Pitts.	St. L.	Pitts.	St. L.	Pitts.	St. L.
New Orleans	.26	.21	.30	.24	.39	.31	.47	.38	.55	.44	.28	.22	.30	.24	.38	.30
Atlanta	.21	21	.24	.24	.31	.31	.38	.38	.44	.44	.22	.22	.24	.24	.30	.30
St. Petersburg	.26	.21	.30	.24	.39	.31	.47	.38	.55	.44	.28	.22	.30	.24	.30	.30
Mobile	.21	.21	.24	.24	.31	.31	.38	.38	.44	.44	.22	.22	.24	.24	.30	.30
Birmingham	.24	.24	.24	.24	.31	.31	.46	.46	.54	.54	.22	.22	.24	.24	.30	.30
Miami	.26	.26	.30	.30	.39	.39	.38	.47	.44	.55	.22	.28	.24	.30	.30	.38
Baltimore	.16	.26	.18	.30	.23	.39	.29	.47	.33	.55	.16	.28	.18	.30	.22	.38
Philadelphia	.16	.26	.18	.30	.23	.39	.29	.47	.33	.55	.16	.28	.18	.30	.22	.38
Morristown	.16	.26	.18	.30	.23	39	.29	.47	.33	.55	.16	.28	.18	.30	.22	.38
Boston	.21	.31	.24	.36	.31	.47	.38	.56	.44	.66	.22	.34	.24	.36	.30	.46
Pittsburgh	0	.21	0	.24	0	.31	0	38	0	.44	0	.22	0	.24	0	.30
Wilmington	.16	.26	.18	.30	.23	.39	29	.47	.33	.55	.16	.28	.18	.30	.22	.38
Bridgeport	.21	.31	.24	.36	31	.47	.38	.56	.44	.66	.22	.34	.24	.36	.30	.46
Trenton	.16	.26	.18	.30	23	.39	.29	.47	.33	.55	.16	.28	.18	.30	.22	.38
New York	.16	.26	.18	.30	.23	.39	.29	.47	.33	.55	.16	.28	.18	.30	.22	.38

Table 10-9. Standard Cost per Cabinet—Shear, Press, and Assembly
(In dollars)

Cabinet size	W 1530	W 1830	W 2430	W 3030	W 3630	W 2418	W 3018	W 3618
Material (steel packing, hardware)	.500	.600	.750	.850	.950	.600	.650	.700
Standard cost (Shear + Press + Assembly), Pittsburgh	1.525	1.750	2.235	2.475	2.625	1.785	1.923	2.085
Standard cost (Shear + Press + Assembly), St. Louis	1.650	1.805	2.250	2.655	2 800	1.955	2.100	2.250

Table 10-10. Standard Cost per Conveyor Hour
(In dollars)

	Line A	Line B	Line C	Line D	Line E
Direct labor	14 00	11 50	13 00	13.00	15.00
Maintenance	2 00	2 00	4 00	4.00	2 00
Fuel	3.00	3 00	4 50	4 50	3.00
Paint, etc	5 00	4 50	3.00	3.00	4.00
Pack and Ship	8 75	7 00	7 00	7.00	8.50
Total	32 75	28.00	31 50	31 50	32 50

Table 10-11. Standard Costs per Cabinet—Paint and Pack *
(In dollars)

Cabinet size	Pittsburgh		St. Louis		
	Line A	Line B	Line C	Line D	Line E
W 1530	1.365	1.750	2.625	2.625	1.693
W 1830	1.638	2.100	3.150	3.150	2.031
W 2430	2.456	3.150	4.725	4.725	2.031
W 3030	2.456	3.150	4.725	4.725	3.047
W 3630	3.275	4.200	6.300	6.300	3.386
W 2418	1.500	1.925	2.888	2.888	1.693
W 3018	1.638	2.100	3.150	3.150	1.862
W 3618	2.183	2.800	4.200	4.200	2.539

* Table 10-11 is derived from Table 10-10 divided by Table 10-4.

orders scheduled for the period required a total of 5,207.18 conveyor hours. This program is to be compared to the program worked out by LP methods.

STATEMENT OF THE PROBLEM

With the foregoing information as given, the problem to be solved is as follows:

> In which plant should the various cabinets be manufactured in order to meet customer delivery promises with greatest profit (margin) to the Walcab Company, considering *freight* as well as *manufacturing costs?*

SOLVING THE PROBLEM BY THE MODI METHOD

The modi method, discussed in detail in Section II, "Methods," is a rapid and relatively simple computational method. Once problems are set up for solution by the method, the calculations are routine and can be carried out by clerical personnel. These features are particularly valuable when management wants answers frequently so as to make the best adjustments to unforeseen changes in demand or unplanned interruptions to production and to consider other variations to the basic program as a basis for future planning.

The modi method requires that all problem information must be ex-

Table 10-12. Assignment of Customers' Orders—Second Quarter

Pittsburgh

Order	No. cabinets	Cabinets per hour	Line A (in hours)	Line B (in hours)
CSCL-1830	500	20	25.00	
CSAT-3030	500	80/9		56.25
CSBO-3030	1,200	40/3	90.00	
CSPH-3630	100	10	10.00	
CSCL-3630	500	10	50.00	
CSAT-3018	1,500	20	75.00	
CSPH-3018	450	20	22.50	
WGPI-1530	50	24	2.10	
WGPI-3030	200	40/3	15.00	
WGAT-3630	600	10	60.00	
WGDT-3630	500	10	50.00	
WGPI-3630	50	10	5.00	
WGAT-3018	200	20	10.00	
MSMO-3630	4,000	20/3		600.00
MSMO-3018	2,500	20	125.00	
JCDT-1530	600	24	25.00	
JCDT-3030	1,000	40/3	75.00	
EMBR-1530	180	24	7.50	
EMBR-3030	170	40/3	12.75	
EMBA-3030	250	40/3	18.75	
EMTR-3630	2,000	10	200.00	
EMWI-3630	3,600	20/3		540.00
EMBA-2418	720	240/11	33.00	
EMBR-2418	120	240/11	5.50	
EMTR-3018	2,000	40/3		150.00
SCPH-1530	9,600	24	400.00	
SCPH-2418	2,880	240/11	132.00	
ALSP-1530	300	24	12.50	
ALSP-3030	150	40/3	11.25	
Total			1,472.83	1,346.25

St. Louis

Order	No. cabinets	Cabinets per hour	Line C–D (in hours)	Line E (in hours)
CSLA-3030	2,000	20/3	300.00	
CSSTL-3030	2,500	20/3	375.00	
CSHN-1530	600	12	50.00	
CSHN-3030	1,200	20/3	180.00	
CSLA-3630	1,000	5	200.00	
CSSE-3018	500	192/11		28.65
WGSF-1530	100	96/5		5.21
WGNO-1530	650	96/5		33.85
WGDA-1830	500	16		31.25
WGSF-3030	100	32/3		9.38
WGDA-3630	500	48/5		52.10
WGNO-3630	100	48/5		10.42
WGKC-3018	1,400	192/11		80.21
WMDE-1830	2,200	10	220.00	
WMDE-3030	600	32/3		56.25
WMLA-3030	620	32/3		58.13
WMPO-3030	1,100	32/3		103.13
WMDE-3630	1,000	48/5		104.17
WMLA-3630	100	48/5		10.42
WMSA-3630	500	5	100.00	
WMSA-2418	240	96/5		12.50
WMLA-3018	920	192/11		52.71
JCMI-1530	240	96/5		12.50
JCCH-1830	525	16		32.81
JCCH-3030	500	32/3		46.87
JCMI-3630	200	48/5		20.83
JCFW-3630	800	48/5		83.33
ALMO-1530	300	96/5		15.63
ALBI-1530	120	96/5		6.25
ALNO-3030	250	32/3		23.44
ALMO-3030	150	32/3		14.06
ALBI-3030	110	32/3		10.31
ALBI-3630	330	48/5		31.38
ALNO-3018	250	192/11		14.33
Total			1,425.00	963.10

Total. 63 production orders requiring 5,207.18 hours

pressed in terms of a common or standard unit. In the case of the Walcab Company, customer demand, rates of production, available time, and profit margins have to be expressed in terms of a common unit to use the modi method. The unit of measure that permits all information to be related in this problem is a standard conveyor hour.

1. Standard conveyor hour

A Standard Conveyor Hour is the time required to produce a given quantity of a product when manufactured in one conveyor line taken as a standard. For example, in this problem Line A in Pittsburgh is set up as standard and all data relating to demand, profit, rates of production, and available time for the other lines are compared to it. Line A was selected as the standard because all other lines can be compared to it easily and because it is the most effective line on which the cabinets can be run. The various conveyor lines are compared to the standard conveyor line through the medium of an index number.

2. Index numbers

Index Numbers represent the production-rate relationship among the various conveyor lines for the products being run, one conveyor line being used as a standard. They are ratios expressed as decimals. Index numbers are determined by considering the production rate per hour by conveyor for *those parts in the current product mix*. Only the rates of production for those cabinets that will be run in the period being considered should be used as a basis for determining index numbers. This means that the indexes will have to be recomputed when the product mix changes. It is quite possible that the indexes will not change, provided the change in mix is not too great. This, however, can only be determined from experience.

The index numbers for the Walcab Company conveyor lines are determined by comparing the production rates given in Table 10-4. Using Line A, Pittsburgh, as the standard and assigning it a value of 1.00 we can establish index numbers for the other lines that reflect the relative production rate. For example, Line A produces 24 W 1530 cabinets per hour of conveyor operation. Line B, on the other hand, produces only 16 cabinets per hour of conveyor operation. For W 1530 cabinets, Line B is only two-thirds as effective as Line A. Using Line A as a standard of 1.00, we find that the relative efficiency or index for Line B is .667. Considering all products to be scheduled and using Line A as a standard, we find that the index number for Line B is .667, for Line C and Line D .500, and for Line E .80. The index number for Line E is derived from an average of the index numbers for the different cabinet sizes, weighted by the quantities involved in the mix.

3. Customer demand expressed in standard conveyor hours

By use of the rate at which the various cabinet sizes are produced on Line A, customer demand by location and cabinet size is given in Table 10-13.

Table 10-13. Customer Releases Expressed in Standard Conveyor Hours

Customer	Destination	Cabinet size	Number cabinets	Cabinets per standard conveyor hour	Standard conveyor hours	
Central Stores	Houston	W 1530	600	24	25.00	
	Cleveland	W 1830	500	20	25.00	
	Los Angeles	W 3030	2,000	13⅓	150.00	
	Houston	W 3030	1,200	13⅓	90.00	
	Atlanta	W 3030	500	13⅓	37.50	
	St. Louis	W 3030	2,500	13⅓	187.50	
	Boston	W 3030	1,200	13⅓	90.00	
	Los Angeles	W 3630	1,000	10	100.00	
	Philadelphia	W 3630	100	10	10.00	
	Cleveland	W 3630	500	10	50.00	
	Seattle	W 3018	500	20	25.00	
	Atlanta	W 3018	1,500	20	75.00	
	Philadelphia	W 3018	450	20	22.50	
						887.50
Ward Gomery	San Francisco	W 1530	100	24	4.17	
	Pittsburgh	W 1530	50	24	2.08	
	New Orleans	W 1530	650	24	27.08	
	Dallas	W 1830	500	20	25.00	
	San Francisco	W 3030	100	13⅓	7.50	
	Pittsburgh	W 3030	200	13⅓	15.00	
	Dallas	W 3630	500	10	50.00	
	Atlanta	W 3630	600	10	60.00	
	Detroit	W 3630	500	10	50.00	
	Pittsburgh	W 3630	50	10	5.00	
	New Orleans	W 3630	100	10	10.00	
	Atlanta	W 3018	200	20	10.00	
	Kansas City	W 3018	1,400	20	70.00	
						335.83
Morris Supply	Morristown	W 3630	4,000	10	400.00	
	Morristown	W 3018	2,500	20	125.00	
						525.00
Apex Land	St. Petersburg	W 1530	300	24	12.50	
	Mobile	W 1530	300	24	12.50	
	Birmingham	W 1530	120	24	5.00	
	New Orleans	W 3030	250	13⅓	18.75	
	St. Petersburg	W 3030	150	13⅓	11.25	
	Mobile	W 3030	150	13⅓	11.25	
	Birmingham	W 3030	110	13⅓	8.25	
	Birmingham	W 3630	330	10	33.00	
	New Orleans	W 3018	250	20	12.50	
						125.00

Table 10-13. Customer Releases Expressed in Standard Conveyor Hours (*continued*)

Customer	Destination	Cabinet size	Number cabinets	Cabinets per standard conveyor hour	Standard conveyor hours	
Western Mail	Denver	W 1830	2,200	20	110.00	
	Denver	W 3030	600	13 1/3	45.00	
	Los Angeles	W 3030	620	13 1/3	46.50	
	Portland	W 3030	1,100	13 1/3	82.50	
	Denver	W 3630	1,000	10	100.00	
	Los Angeles	W 3630	100	10	10.00	
	Salem	W 3630	500	10	50.00	
	Salem	W 2418	240	21 9/11	11.00	
	Los Angeles	W 3018	920	20	46.00	
						501.00
Jensen Company	Minneapolis	W 1530	240	24	10.00	
	Detroit	W 1530	600	24	25.00	
	Chicago	W 1830	525	20	26.25	
	Detroit	W 3030	1,000	13 1/3	75.00	
	Chicago	W 3030	500	13 1/3	37.50	
	Minneapolis	W 3630	200	10	20.00	
	Fort Wayne	W 3630	800	10	80.00	
						273.75
Eastern Mail	Bridgeport	W 1530	180	24	7.50	
	Bridgeport	W 3030	170	13 1/3	12.75	
	Baltimore	W 3030	250	13 1/3	18.75	
	Trenton	W 3630	2,000	10	200.00	
	Wilmington	W 3630	3,600	10	360.00	
	Baltimore	W 2418	720	21 9/11	33.00	
	Bridgeport	W 2418	120	21 9/11	5.50	
	Trenton	W 3018	2,000	20	100.00	
						737.50
Stacey Construction	Philadelphia	W 1530	9,600	24	400.00	
	Philadelphia	W 2418	2,880	21 9/11	132.00	
						532.00
Total standard hours required						3,917.58

4. Available standard conveyor hours

The available Standard Conveyor Hours for the period are determined by multiplying the adjusted available time on each line by its index number. The standard conveyor hours by conveyor line for the period are given in Table 10-14.

To clarify the meaning of these standard capacities, it may be convenient to think of the standard hour as a unit of production rather than

Table 10-14. Available Standard Conveyor Hours

Line	Maximum time available per quarter (in hours)	Per cent operator time	Adjusted available time (in hours)	Index	Standard conveyor hours
A	1,560	96 0	1,497.60	1.000	1,497.60
B	1,560	87.5	1,365.00	.667	910.00
C	1,560	85.5	1,333 80	.500	666.90
D	1,560	95 00	1,482.00	.500	741.00
E	1,560	89.3	1,393.08	.800	1,114.46
Total					4,929.96

a unit of time. For example, Line A has 1,497.60 hours available, and in this time it can produce 1,497.60 standard hours' worth of cabinets. Line B, on the other hand, has 1,365.00 hours of time available. But in this time it can produce only 910.00 (1365 × .667) standard hours' worth of cabinets.

5. Profit per standard conveyor hour

The profit margin obtained by making a certain cabinet on one of the lines can be calculated from the data in the tables. Table 10-15 shows how this is done. The standard costs and freight are subtracted from the selling price to get the profit margin per cabinet.

The company which originally made this application used a direct-costing system which separated fixed costs from costs which could be directly allocated to products. The data in the Walcab Company problem represents the same kind of cost accounting. The standard costs shown do not contain the fixed portion.

The profit margins shown are therefore larger than the actual net profit. They are, however, directly comparable to one another. Since the LP calculation deals with the differences between the various margins, the inclusion of fixed costs in the margins will not affect the solution.

The total profit shown for any one program will also contain fixed costs and will appear artificially large. The difference in profit between two programs, however, will be the actual difference in net profit.

To use the profit values in the modi method, they must be expressed in terms of the standard unit—the standard conveyor hours. This is done by multiplying the profit per piece by the number of pieces that can be produced on the standard machine in one hour. The fact that use of the

Table 10-15. Profit per Standard Conveyor Hour
(In dollars)
Walcab Company

Customer	Destination	Cabinet size	Plant line	Selling price	Standard cost			Freight equalization	Cost per cabinet	Margin per cabinet	Cabinets per standard conveyor hour	Margin per standard conveyor hour
					Material	Shear, Press, and Assembly	Paint and Pack					
Central Stores	Houston	W 1530	A	6.500	.500	1.525	1.365	.360	3.750	2.750	24	66.000 *
			B	6.500	.500	1.525	1.750	.360	4.135	2.365	24	56.760
			C or D	6.500	.500	1.650	2.625	.260	5.035	1.465	24	35.160
			E	6.500	.500	1.650	1.693	.260	4.103	2.397	24	57.528
		W 3030	A	11.270	.850	2.475	2.456	.770	6.551	4.719	40/3	62.920 *
			B	11.270	.850	2.475	3.150	.770	7.245	4.025	40/3	53.667
			C or D	11.270	.850	2.655	4.725	.550	8.780	2.490	40/3	33.200 *
			E	11.270	.850	2.655	3.047	.550	7.102	4.168	40/3	55.573
	Cleveland	W 1830	A	7.700	.600	1.750	1.638	.220	4.208	3.492	20	69.840
			B	7 700	.600	1.750	2.100	.220	4.670	3.030	20	60.600 *
			C or D	7.700	.600	1.805	3.150	.340	5.895	1.805	20	36.100
			E	7.700	.600	1.805	2.031	.340	4.776	2.924	20	58.480
		W 3630	A	13.830	.950	2.625	3.275	.410	7.260	6.570	10	65.700
			B	13.830	.950	2.625	4.200	.410	8.185	5.645	10	56.450
			C or D	13.830	.950	2.800	6.300	.560	10.610	3.220	10	32.200
			E	13.830	.950	2.800	3.386	.560	7.696	6.134	10	61.340 *
	Los Angeles	W 3030	A	11.270	.850	2.475	2.456	.740	6.521	4.749	40/3	63 320
			B	11.270	.850	2.475	3.150	.740	7.215	4 055	40/3	54 067
			C or D	11.270	.850	2.655	4.725	.650	8.880	2.390	40/3	31 867
			E	11.270	.850	2.655	3.047	.650	7.207	4.068	40/3	52.240
		W 3630	A	13.830	.950	2.625	3.275	.880	7.730	6.100	10	61.000
			B	13.830	.950	2.625	4.200	.880	8.655	5.175	10	51.750
			C or D	13.830	.950	2.800	6.300	.770	10.820	3.010	10	30.100
			E	13.830	.950	2.800	3.386	.770	7.906	5 924	10	59 240 *
	Atlanta	W 3030	A	11.270	.850	2.475	2.456	.380	6.161	5.109	40/3	68.120 *
			B	11.270	.850	2.475	3.150	.380	6.855	4.415	40/3	58.867
			C or D	11.270	.850	2.655	4.725	.380	8.610	2.660	40/3	35.467
			E	11.270	.850	2.655	3.047	.380	6.932	4.338	40/3	57.840
		W 3018	A	8.000	.650	1.923	1.638	.240	4.451	3.549	20	70.980
			B	8.000	.650	1.923	2.100	.240	4.913	3.087	20	61.740 *
			C or D	8.000	.650	2.100	3.150	.240	6.140	1.860	20	37.200
			E	8.000	.650	2.100	1.862	.240	4.852	3.148	20	62 960
	St. Louis	W 3030	A	11.270	.850	2.475	2.456	.380	6.161	5.109	40/3	68.120
			B	11.270	.850	2.475	3.150	.380	6.855	4.415	40/3	58.867
			C or D	11.270	.850	2 655	4.725	.000	8.230	3.040	40/3	40.533 *
			E	11.270	.850	2.655	3.047	.000	6.552	4.718	40/3	62.907
	Boston	W 3030	A	11.270	.850	2.475	2.456	.380	6.161	5.109	40/3	68.120 *
			B	11.270	.850	2.475	3.150	.380	6.855	4.415	40/3	58.867
			C or D	11.270	.850	2.655	4.725	.560	8.790	2.480	40/3	33.067
			E	11.270	.850	2.655	3.047	.560	7.112	4.158	40/3	55.440

* LP assignment calculated in final matrix—most profitable program.

Table 10-15. Profit per Standard Conveyor Hour (in dollars): Walcab Company (*continued*)

Customer	Destination	Cabinet size	Plant line	Selling price	Standard cost: Material	Standard cost: Shear, Press, and Assembly	Standard cost: Paint and Pack	Freight equalization	Cost per cabinet	Margin per cabinet	Cabinets per standard conveyor hour	Margin per standard conveyor hour
Central Stores (*continued*)	Philadelphia	W 3630	A	13.830	.950	2.625	3.275	.330	7.180	6.650	10	66.500
			B	13.830	.950	2.625	4.200	.330	8.105	5.725	10	57.250
			C or D	13.830	.950	2.800	6.300	.550	10.600	3.230	10	32.300
			E	13.830	.950	2.800	3.386	.550	7.686	6.144	10	61.440 *
		W 3018	A	8.000	.650	1.923	1.638	.180	4.391	3.609	20	72.180
			B	8.000	.650	1.923	2.100	.180	4.853	3.147	20	62.940 *
			C or D	8.000	.650	2.100	3.150	.300	6.200	1.800	20	36.000
			E	8.000	.650	2.100	1.862	.300	4.912	3.088	20	61.760
	Seattle	W 3018	A	8.000	.650	1.923	1.638	.480	4.691	3.309	20	66.180 *
			B	8.000	.650	1.923	2.100	.480	5.153	2.847	20	56.940
			C or D	8.000	.650	2.100	3.150	.420	6.320	1.680	20	33.600
			E	8.000	.650	2.100	1.862	.420	5.032	2.968	20	59.360
Ward Gomery	San Francisco	W 1530	A	6.500	.500	1.525	1.365	.410	3.800	2.700	24	64.800 *
			B	6.500	.500	1.525	1.750	.410	4.185	2.315	24	55.560
			C or D	6.500	.500	1.650	2.625	.360	5.135	1.365	24	32.760
			E	6.500	.500	1.650	1.693	.360	4.203	2.297	24	55.128
		W 3030	A	11.270	.850	2.475	2.456	.740	6.521	4.749	40/3	63.320 *
			B	11.270	.850	2.475	3.150	.740	7.215	4.055	40/3	54.067
			C or D	11.270	.850	2.655	4.725	.650	8.880	2.390	40/3	31.867
			E	11.270	.850	2.655	3.047	.650	7.202	4.068	40/3	54.240
	Pittsburgh	W 1530	A	6.500	.500	1.525	1.365	.000	3.390	3.110	24	74.640 *
			B	6.500	.500	1.525	1.750	.000	3.775	2.725	24	65.400
			C or D	6.500	.500	1.650	2.625	.210	4.985	1.515	24	36.360
			E	6.500	.500	1.650	1.693	.210	4.053	2.447	24	58.728
		W 3030	A	11.270	.850	2.475	2.456	.000	5.781	5.489	40/3	73.187 *
			B	11.270	.850	2.475	3.150	.000	6.475	4.795	40/3	63.933
			C or D	11.270	.850	2.655	4.725	.380	8.610	2.660	40/3	35.467
			E	11.270	.850	2.655	3.047	.380	6.932	4.338	40/3	57.840
		W 3630	A	13.830	.950	2.625	3.275	.000	6.850	6.980	10	69.800 *
			B	13.830	.950	2.625	4.200	.000	7.775	6.055	10	60.550
			C or D	13.830	.950	2.800	6.300	.440	10.490	3.340	10	33.400
			E	13.830	.950	2.800	3.386	.440	7.576	6.254	10	62.540
	New Orleans	W 1530	A	6.500	.500	1.525	1.365	.260	3.650	2.850	24	68.400 *
			B	6.500	.500	1.525	1.750	.260	4.035	2.465	24	59.160
			C or D	6.500	.500	1.650	2.625	.210	4.985	1.515	24	36.360
			E	6.500	.500	1.650	1.693	.210	4.053	2.447	24	58.728
		W 3630	A	13.830	.950	2.625	3.275	.550	7.400	6.430	10	64.300
			B	13.830	.950	2.625	4.200	.550	8.325	5.505	10	55.050
			C or D	13.830	.950	2.800	6.300	.440	10.490	3.340	10	33.400
			E	13.830	.950	2.800	3.386	.440	7.576	6.254	10	62.540 *
	Dallas	W 1830	A	7.700	.600	1.750	1.638	.420	4.408	3.292	20	65.840
			B	7.700	.600	1.750	2.100	.420	4.870	2.830	20	56.600
			C or D	7.700	.600	1.805	3.150	.300	5.855	1.845	20	36.900 *
			E	7.700	.600	1.805	2.031	.300	4.736	2.964	20	59.280

***Table* 10-15. Profit per Standard Conveyor Hour (in dollars): Walcab Company (*continued*)**

Customer	Destination	Cabinet size	Plant line	Selling price	Standard cost			Freight equalization	Cost per cabinet	Margin per cabinet	Cabinets per standard conveyor hour	Margin per standard conveyor hour
					Material	Shear, Press, and Assembly	Paint and Pack					
Ward Gomery (*continued*)	Dallas (*continued*)	W 3630	A	13.830	.950	2.625	3.275	.770	7.620	6.210	10	62.100
			B	13.830	.950	2.625	4.200	.770	8.545	5.285	10	52.850
			C or D	13.830	.950	2.800	6.300	.550	10.600	3.230	10	32.300
			E	13.830	.950	2.800	3.386	.550	7.686	6.144	10	61.440 *
	Atlanta	W 3630	A	13.830	.950	2.625	3.275	.440	7.290	6.540	10	65.400
			B	13.830	.950	2.625	4.200	.440	8.215	5.615	10	56.150
			C or D	13.830	.950	2.800	6.300	.440	10.490	3.340	10	33.400
			E	13.830	.950	2.800	3.386	.440	7.576	6.254	10	62.540 *
		W 3018	A	8.000	.650	1.923	1.638	.240	4.451	3.549	20	70.980
			B	8.000	.650	1.923	2.100	.240	4.913	3.087	20	61.740 *
			C or D	8.000	.650	2.100	3.150	.240	6.140	1.860	20	37.200
			E	8 000	650	2.100	1.862	.240	4.852	3.148	20	62.960
	Detroit	W 3630	A	13.830	.950	2.625	3.275	.380	7.230	6.600	10	66.000
			B	13.830	.950	2.625	4.200	.380	8.155	5.675	10	56.750
			C or D	13.830	950	2.800	6.300	.530	10.580	3.250	10	32.500
			E	13.830	.950	2.800	3.386	.530	7.666	6.164	10	61.640 *
	Kansas City	W 3018	A	8.000	.650	1.923	1.638	.180	4.391	3.609	20	72.180 *
			B	8.000	.650	1.923	2.100	.180	4.853	3.147	20	62.940 *
			C or D	8.000	.650	2.100	3.150	.240	6.140	1.860	20	37.200
			E	8.000	.650	2.100	1.862	.240	4.852	3.148	20	62.960
Morris Supply	Morristown	W 3630	A	12.500	.950	2.625	3.275	.330	7.180	5.320	10	53.200
			B	12.500	.950	2.625	4.200	.330	8.105	4.395	10	43 950
			C or D	12.500	950	2.800	6.300	.550	10.600	1.900	10	19.000
			E	12.500	.950	2.800	3.386	.550	7.686	4.814	10	48.140 *
		W 3018	A	7.200	.650	1.923	1.638	.180	4.391	2.809	20	56.180
			B	7.200	.650	1.923	2.100	.180	4.853	2.347	20	46.940 *
			C or D	7.200	.650	2.100	3.150	.300	6.200	1.000	20	20.000
			E	7.200	.650	2.100	1.862	.300	4.912	2.288	20	45.760
Apex Land	St. Petersburg	W 1530	A	6.500	.500	1.525	1.365	.260	3.650	2.850	24	68.400 *
			B	6.500	.500	1.525	1.750	.260	4.035	2.465	24	59.160
			C or D	6.500	.500	1.650	2.625	.210	4.985	1.515	24	36.360
			E	6.500	.500	1.650	1.693	.210	4.053	2.447	24	58.728
		W 3030	A	11.270	.850	2.475	2.456	.470	6.251	5.019	40/3	66.920 *
			B	11.270	.850	2.475	3.150	.470	6.945	4.325	40/3	57.667
			C or D	11.270	.850	2.655	4.725	.380	8.610	2.660	40/3	35.467
			E	11.270	.850	2.655	3.047	.380	6.932	4.338	40/3	57.840
	Mobile	W 1530	A	6.500	.500	1.525	1.365	.210	3.600	2.900	24	69.600 *
			B	6.500	.500	1.525	1.750	.210	3.985	2.515	24	60.360
			C or D	6.500	.500	1.650	2.625	.210	4.985	1.515	24	36.360
			E	6.500	.500	1.650	1.693	.210	4.053	2.447	24	58.728
		W 3030	A	11.270	.850	2.475	2.456	.380	6.161	5.109	40/3	68.120 *
			B	11.270	.850	2.475	3.150	.380	6.855	4.415	40/3	58.867
			C or D	11.270	.850	2.655	4.725	.380	8.610	2.660	40/3	35.467
			E	11.270	.850	2.655	3.047	.380	6.932	4.338	40/3	57.840

Table 10-15. Profit per Standard Conveyor Hour (in dollars): Walcab Company (*continued*)

Customer	Destination	Cabinet size	Plant line	Selling price	Standard cost			Freight equalization	Cost per cabinet	Margin per cabinet	Cabinets per standard conveyor hour	Margin per standard conveyor hour
					Material	Shear, Press, and Assembly	Paint and Pack					
Apex Land (*continued*)	Birmingham	W 1530	A	6.500	.500	1.525	1.365	.240	3.630	2.870	24	68.880 *
			B	6.500	.500	1.525	1.750	.240	4.015	2.485	24	59.640
			C or D	6.500	.500	1.650	2.625	.240	5.015	1.485	24	35.640
			E	6.500	.500	1.650	1.693	.240	4.083	2.417	24	58.008
		W 3030	A	11.270	.850	2.475	2.456	.460	6.241	5.029	40/3	67.053 *
			B	11.270	.850	2.475	3.150	.460	6.935	4.335	40/3	57.800
			C or D	11.270	.850	2.655	4.725	.460	8.690	2.580	40/3	34.400
			E	11.270	.850	2.655	3.047	.460	7.012	4.258	40/3	56.773
		W 3630	A	13.830	.950	2.625	3.275	.540	7.390	6.440	10	64.400
			B	13.830	.950	2.625	4.200	.540	8.315	5.515	10	55.150
			C or D	13.830	.950	2.800	6.300	.540	10.590	3.240	10	32.400
			E	13.830	.950	2.800	3.386	.540	7.676	6.154	10	61.540 *
	New Orleans	W 3030	A	11.270	.850	2.475	2.456	.470	6.251	5.019	40/3	66.920 *
			B	11.270	.850	2.475	3.150	.470	6.945	4.325	40/3	57.667
			C or D	11.270	.850	2.655	4.725	.380	8.610	2.660	40/3	35.467
			E	11.270	.850	2.655	3.047	.380	6.932	4.338	40/3	57.840
		W 3018	A	8.000	.650	1.923	1.638	.300	4.511	3.489	20	69.780
			B	8.000	.650	1.923	2.100	.300	4.973	3.027	20	60.540 *
			C or D	8.000	.650	2.100	3.150	.240	6.140	1.860	20	37.200
			E	8.000	.650	2.100	1.862	.240	4.852	3.148	20	62.960
Western Mail	Denver	W 1830	A	7.700	.600	1.750	1.638	.420	4.408	3.292	20	65.840
			B	7.700	.600	1.750	2.100	.420	4.870	2.830	20	56.600
			C or D	7.700	.600	1.805	3.150	.300	5.855	1.845	20	36.900 *
			E	7.700	600	1.805	2.031	.300	4.736	2.964	20	59.280
		W 3030	A	11.270	.850	2.475	2.456	.650	6.440	4.830	40/3	64.400
			B	11 270	.850	2.475	3 150	.650	7.125	4.145	40/3	55.267 *
			C or D	11.270	.850	2.655	4.725	.470	8.700	2.570	40/3	34.267
			E	11.270	850	2.655	3.047	.470	7.022	4.248	40/3	56.640
		W 3630	A	13.830	950	2.625	3.275	.770	7.620	6.210	10	62.100
			B	13.830	.950	2.625	4.200	.770	8.545	5.285	10	52.850
			C or D	13.830	.950	2.800	6.300	.550	10.600	3.230	10	32.300
			E	13.830	.950	2.800	3.386	.550	7.686	6.144	10	61.440 *
	Los Angeles	W 3030	A	11.270	.850	2.475	2.456	.740	6.521	4.749	40/3	63.320 *
			B	11.270	.850	2.475	3.150	.740	7.215	4.055	40/3	54.067
			C or D	11.270	.850	2.655	4.725	.650	8.880	2.390	40/3	31.867
			E	11.270	.850	2.655	3.047	.650	7.202	4.068	40/3	54.240
		W 3630	A	13.830	.950	2.625	3.275	.880	7.730	6.100	10	61.000
			B	13.830	.950	2.625	4.200	.880	8.655	5.175	10	51 750
			C or D	13.830	.950	2.800	6.300	.770	10.820	3.010	10	30.100
			E	13.830	.950	2.800	3.386	.770	7.906	5.924	10	59.240 *
		W 3018	A	8.000	.650	1.923	1.638	.480	4.691	3.309	20	66.180 *
			B	8.000	.650	1.923	2.100	.480	5.153	2.847	20	56.940
			C or D	8.000	.650	2.100	3.150	.420	6.320	1.680	20	33.600
			E	8.000	.650	2.100	1.862	.420	5.032	2.968	20	59.360

***Table* 10-15. Profit per Standard Conveyor Hour (in dollars): Walcab Company (*continued*)**

Customer	Destination	Cabinet size	Plant line	Selling price	Standard cost			Freight equalization	Cost per cabinet	Margin per cabinet	Cabinets per standard conveyor hour	Margin per standard conveyor hour
					Material	Shear, Press, and Assembly	Paint and Pack					
Western Mail (*continued*)	Portland	W 3030	A	11.270	.850	2.475	2.456	.740	6.521	4.749	40/3	63.320 *
			B	11.270	.850	2.475	3.150	.740	7.215	4.055	40/3	54.067
			C or D	11.270	.850	2.655	4.725	.650	8.880	2.390	40/3	31.867
			E	11.270	.850	2.655	3.047	.650	7.202	4.068	40/3	54.240
	Salem	W 3630	A	13.830	.950	2.625	3.275	.880	7.730	6.100	10	61.000
			B	13.830	.950	2.625	4.200	.880	8.655	5.175	10	51.750
			C or D	13.830	.950	2.800	6.300	.770	10.820	3.010	10	30.100
			E	13.830	.950	2.800	3.386	.770	7.906	5.924	10	59.240 *
		W 2418	A	7.380	.600	1.785	1.500	.460	4.345	3.035	240/11	66.218 *
			B	7.380	.600	1.785	1.925	.460	4.770	2.610	240/11	56.945
			C or D	7.380	.600	1.955	2.888	.400	5.843	1.537	240/11	33.527
			E	7.380	.600	1.955	1.693	.400	4.648	2.732	240/11	59.607
Stacey Construction	Philadelphia	W 1530	A	6.000	.500	1.525	1 365	.160	3.550	2.450	24	58.800
			B	6 000	500	1.525	1.750	.160	3.935	2.065	24	49.560 *
			C or D	6 000	.500	1 650	2.625	.260	5.035	.965	24	23.160
			E	6 000	.500	1.650	1.693	.260	4.103	1.897	24	45.528
		W 2418	A	6.750	.600	1.785	1.500	.160	4.045	2.705	240/11	59.018 *
			B	6.750	.600	1.785	1.925	.160	4.470	2.280	240/11	49.745
			C or D	6.750	.600	1.955	2.888	280	5.723	1.027	240/11	22.407
			E	6.750	.600	1.955	1.693	.280	4.528	2.222	240/11	48.480
Jensen Company	Minneapolis	W 1530	A	6.500	.500	1.525	1.365	.260	3.650	2.850	24	68.400 *
			B	6 500	.500	1.525	1.750	.260	4.035	2.465	24	59.160
			C or D	6.500	.500	1.650	2.625	.210	4.985	1.515	24	36 360
			E	6.500	.500	1.650	1.693	.210	4.053	2.447	24	58.728
		W 3630	A	13.830	.950	2.625	3.275	.550	7.400	6.430	10	64.300
			B	13.830	.950	2.625	4.200	.550	8.325	5.505	10	55.050
			C or D	13.830	.950	2.800	6.300	.440	10.490	3.340	10	33.400
			E	13.830	.950	2.800	3.386	.440	7.576	6.254	10	62.540 *
	Detroit	W 1530	A	6 500	.500	1.525	1.365	.180	3.570	2.930	24	70.320
			B	6.500	.500	1.525	1.750	.180	3.955	2.545	24	61 080 *
			C or D	6.500	500	1 650	2.625	.250	5.025	1.475	24	35.400
			E	6 500	.500	1.650	1.693	.250	4.093	2.407	24	57.768
		W 3030	A	11.270	.850	2.475	2.456	360	6.141	5 129	40/3	68.387 *
			B	11.270	.850	2.475	3.150	.360	6.835	4.435	40/3	59.133
			C or D	11.270	.850	2.655	4.725	.470	8.700	2.570	40/3	34.267
			E	11.270	.850	2.655	3.047	.470	7.022	4.248	40/3	56.640
	Chicago	W 1830	A	7.700	.600	1.750	1.638	.200	4.188	3.512	20	70.240 *
			B	7.700	.600	1.750	2.100	.200	4.650	3.050	20	61.000
			C or D	7.700	.600	1.805	3.150	.210	5.765	1.935	20	38.700
			E	7.700	.600	1.805	2 031	.210	4 646	3.054	20	61.080
		W 3030	A	11.270	.850	2.475	2.456	.360	6.141	5.129	40/3	68.387 *
			B	11.270	.850	2.475	3.150	360	6.835	4.435	40/3	59.133
			C or D	11.270	.850	2.655	4.725	.380	8.610	2.660	40/3	35.467
			E	11.270	.850	2.655	3.047	.380	6.932	4.338	40/3	57.840

Table 10-15. Profit per Standard Conveyor Hour (in dollars): Walcab Company (*continued*)

Customer	Destination	Cabinet size	Plant line	Selling price	Standard cost			Freight equalization	Cost per cabinet	Margin per cabinet	Cabinets per standard conveyor hour	Margin per standard conveyor hour
					Material	Shear, Press, and Assembly	Paint and Pack					
Jensen Company (*continued*)	Fort Wayne	W 3630	A	13.830	.950	2.625	3.275	.330	7.180	6.650	10	66.500
			B	13.830	.950	2.625	4.200	.330	8.105	5.725	10	57.250
			C or D	13.830	.950	2.800	6.300	.330	10.380	3.450	10	34.500
			E	13.830	.950	2.800	3.386	.330	7.466	6.364	10	63.640 *
Eastern Mail	Bridgeport	W 1530	A	6.500	.500	1.525	1.365	.210	3.600	2.900	24	69.600
			B	6.500	.500	1.525	1.750	.210	3.985	2.515	24	60.360 *
			C or D	6.500	.500	1.650	2.625	.310	5.085	1.415	24	33.960
			E	6.500	.500	1.650	1.693	.310	4.153	2.347	24	56.328
		W 3030	A	11.270	.850	2.475	2.456	.380	6.161	5.109	40/3	68.120 *
			B	11.270	.850	2.475	3.150	.380	6.855	4.415	40/3	58.867
			C or D	11.270	.850	2.655	4.725	.560	8.790	2.480	40/3	33.067
			E	11.270	.850	2.655	3.047	.560	7.112	4.158	40/3	55.440
		W 2418	A	7.380	.600	1.785	1.500	.220	4.105	3.275	240/11	71.455 *
			B	7.380	.600	1.785	1.925	.220	4.530	2.850	240/11	62.182
			C or D	7.380	.600	1.955	2.888	.340	5.783	1.597	240/11	34.843
			E	7.380	600	1.955	1.693	.340	4.588	2.792	240/11	60.916
	Baltimore	W 3030	A	11.270	.850	2.475	2.456	.290	6.071	5.199	40/3	69.320 *
			B	11.270	.850	2.475	3.150	.290	6.765	4.505	40/3	60.067
			C or D	11.270	.850	2.655	4.725	.470	8.700	2.570	40/3	34.267
			E	11.270	.850	2.655	3.047	.470	7.022	4.248	40/3	56.640
		W 2418	A	7.380	.600	1.785	1.500	.160	4.045	3.335	240/11	72.745 *
			B	7.380	.600	1.785	1.925	.160	4.470	2.910	240/11	63.941
			C or D	7.380	.600	1.955	2.888	.280	5.723	1.657	240/11	36.153
			E	7.380	600	1.955	1.693	.280	4.528	2.852	240/11	62.225
	Trenton	W 3630	A	13.830	.950	2.625	3.275	.330	7.180	6.650	10	66.500 *
			B	13.830	.950	2.625	4.200	.330	8.105	5.725	10	57.250
			C or D	13.830	.950	2.800	6.300	.550	10.600	3.230	10	32.300
			E	13.830	.950	2.800	3.386	.550	7.686	6.144	10	61.440 *
		W 3018	A	8.000	.650	1.923	1.638	.180	4.391	3.609	20	72.180
			B	8.000	.650	1.923	2.100	.180	4.853	3.147	20	62.940 *
			C or D	8.000	.650	2.100	3.150	.300	6.200	1.800	20	36.000
			E	8.000	.650	2.100	1.862	.300	4.912	3.088	20	61.760
	Wilmington	W 3630	A	13.830	.950	2.625	3.275	.330	7.180	6.650	10	66.500 *
			B	13.830	.950	2.625	4.200	.330	8.105	5.725	10	57.250
			C or D	13.830	.950	2.800	6.300	.550	10.600	3.230	10	32.300
			E	13.830	.950	2.800	3.386	.550	7.686	6.144	10	61.440

modi compels development of profit data in terms of the problem measure has been of great value in providing proper perspective. Cost figures are rarely if ever available in these terms unless they have been computed for use in linear programming. In this case, the profits per piece of various units may give some idea of the relative desirability of selling certain products, but they do not help in establishing production programs. The production problem involves allocating machine capacity. When profits are expressed in terms of machine hours, the profitability of allocating production time to various products is directly comparable.

The profit per standard conveyor hour for each product by destination by the conveyor line on which it can be produced is given in Table 10-15. The asterisks indicate the profit margin per standard conveyor hour for the individual orders as run in the most profitable program.

MOST PROFITABLE ASSIGNMENT OF ORDERS TO THE TWO PLANTS

Once the pertinent problem information has been expressed in standard conveyor hours, it is then set up in a table, or modi matrix. Table 10-16 in pocket at back of the book shows the most profitable program, including where and in what amounts each order should be produced. The most profitable program indicates that a profit of $226,179.10 is possible if production is planned as indicated by the circled values.

A breakdown of the most profitable load by customer and cabinet size is given in Tables 10-18 and 10-19 at the end of this chapter.

COMPARISON OF ACTUAL AND LP PROGRAMS

A summary of the differences between the actual and the LP programs that are significant for management planning is given in Table 10-17. The table shows an increase in profits of $11,121.92, and a reduction of 160.46 conveyor hours is possible if management uses LP information as a basis for assigning production orders to the two plants. This can be achieved through better planning and does not require an additional investment in facilities or manpower. It can be obtained from existing resources.

Most of the differences between the actual and the LP programs are the result of *not* shipping from the nearest plant according to policy and commercial commitment. A check of the differences between programs in the assignment of orders to the two plants (Table 10-18) shows that the LP program included many cross shipments from east to west and west to east. For example, the 100 W 3030 cabinets for Ward Gomery, San Francisco, are produced and shipped from Pittsburgh rather than

Table 10-17. Comparison of Actual and LP Programs

	Profit (in dollars)	Conveyor hours required	Assignment of capacity by conveyor line (in hours)				Average profit per cabinet (in dollars)
			A	B	C, D	E	
LP program	226,179.10	5,046.72	1,497.60	1,365.00	791.04	1,393.08	3.858
Actual program	215,057.18	5,207.18	1,472.83	1,346.25	1,425 00	963.10	3.669
Difference	+11,121.92	−160 46	+24.77	+18.75	−633.96	+429.98	+.189

from St. Louis, which is 800 miles closer to San Francisco. The same is true for the requirement of Central Stores, Los Angeles; Western Mail, Denver; Jensen Company, Minneapolis; and many others.

The reason for this is that the linear-programming technique compares the rate at which profit can be made on each of the products on each of the lines. These profits include the freight costs involved to each destination. The LP calculation then assigns the orders within the available capacity of the various production lines according to highest profit for the *mix*.

The allocation or assignment of capacity to orders on this basis causes some shipments to be made from west to east. For example, the requirement of Morris Supply, Morristown, New Jersey, for 4,000 W 3630 cabinets is supplied from St. Louis instead of from Pittsburgh. The same situation is true for Central Stores, Philadelphia; and Eastern Mail, Trenton, New Jersey. The total gain in profit from following this program exceeds the increased costs incurred by a few individual orders.

The LP program also confirms that either Line C or Line D can be closed down completely if the present level of demand continues for any length of time. Should management decide, however, that they want to continue to operate Line C and Line D at some level of capacity for reasons of personnel policy or anticipated sales, the LP technique will provide management with a program that will accomplish the desired result at maximum profit under the new policy. The difference between the profit under each set of circumstances places a dollar sign on the desirability of following the policy. The profit difference is also one measure of the money that the firm can afford to invest in new equip-

Table **10-18. Differences in Assignments of Customers' Orders (41 orders run differently)**

Customer	Cabinet size	Destination	Best assignment *	Actual assignment *	Best profit (in dollars)	Actual profit (in dollars)	Difference (in dollars)
Central Stores	W 1530	Houston	P-A	S-C	1,650 00	879.00	771.00
	W 1830	Cleveland	P-B	P-A	1,515.00	1,746.00	−231.00
	W 3030	Los Angeles	P-A	S-C	9,498.00	4,780.00	4,718.00
	W 3030	Houston	P-A		1,071.24		
	W 3030	Houston	S-C	S-C	2,422.77	2,988.00	505.98
	W 3030	Atlanta	P-A	P-B	2,554.50	2,207.50	347.00
	W 3630	Los Angeles	S-E	S-C	5,924.00	3,010.00	2,914.00
	W 3630	Philadelphia	S-E	P-A	614.40	665.00	−50.60
	W 3630	Cleveland	S-E	P-A	3,067.00	3,285.00	−218.00
	W 3018	Seattle	P-A	S-E	1,654.50	1,484.00	170.50
	W 3018	Atlanta	P-B	P-A	4,630.50	5,323.50	−693.00
	W 3018	Philadelphia	P-B	P-A	1,416.15	1,624.05	−207.90
Ward Gomery	W 1530	San Francisco	P-A	S-E	270.00	229.70	40.30
	W 1530	New Orleans	P-A	S-E	1,852.50	1,590.55	261.95
	W 1830	Dallas	S-C	S-E	922.50	1,482.00	−559.50
	W 3030	San Francisco	P-A	S-E	474.90	406.80	68.10
	W 3630	Atlanta	S-E	P-A	3,752.40	3,924.00	171.60
	W 3630	Detroit	S-E	P-A	3,082.00	3,300.00	218.00
	W 3018	Atlanta	P-B	P-A	617.40	709.80	92.40
	W 3018	Kansas City	P-B		3,933.75		101.63
			P-A	S-E	541 35	4,407.20	67.90
Morris Supply	W 3630	Morristown	S-E	P-B	19,256 00	17,580 00	1,676.00
	W 3018	Morristown	P-B	P-A	5,867.50	7,022.50	−1,155.00
Apex Land	W 1530	Mobile	P-A	S-E	870.00	734.10	135.90
	W 1530	Birmingham	P-A	S-E	341.40	290.04	54.36
	W 3030	New Orleans	P-A	S-E	1,274.75	1,084.50	170.25
	W 3030	Mobile	P-A	S-E	766 35	650.70	105.65
	W 3030	Birmingham	P-A	S-E	553.19	468.38	84.81
	W 3018	New Orleans	P-B	S-E	756.75	787.00	−30.25
Western Mail	W 3030	Denver	P-B	S-E	2,487.00	2,548.80	−61.80
	W 3030	Los Angeles	P-A	S-E	2,944.38	2,522.16	422.22
	W 3030	Portland	P-A	S-E	5,223.90	4,474.80	749.10
	W 3630	Salem	S-E	S-C	2,962.00	1,505.50	1,456.50
	W 2418	Salem	P-A	S-E	728.40	655.68	72.72
	W 3018	Los Angeles	P-A	S-E	3,044.28	2,730.56	313.72
Jensen Company	W 1530	Minneapolis	P-A	S-E	684.00	587.28	96.72
	W 1530	Detroit	P-B	P-A	1,527.00	1,758.00	−231.00
	W 1830	Chicago	P-A	S-E	1,843.80	1,603.35	240.45
	W 3030	Chicago	P-A	S-E	2,564.50	2,169.00	395.50
Eastern Mail	W 1530	Bridgeport	P-B	P-A	452.70	522.00	−69.30
	W 3630	Trenton	S-E	P-A	5,621.76	13,300.00	−462.99
	W 3630	Trenton	P-A		7,215.25		
	W 3630	Wilmington	P-A	P-B	23,940.00	20,610.00	3,330.00
Stacey Construction	W 1530	Philadelphia	P-B	P-A	19,824.00	23,520.00	−3,696.00
Total							11,121.92

***P = Pittsburgh S = St. Louis A, B, C, and E : Production Lines.**

ment and methods to raise the efficiency of Line C and Line D at St. Louis.

Once planning management has installed the LP technique it is possible to provide them with information to guide their decisions on the following:

1. The best revised program should changes in demand or interruption to production occur
2. A program of planned preventive maintenance
3. Alternate programs of freight costs and price concessions to be competitive in distant markets

Table 10-19. Orders Run Same in Both Programs
(22 orders same in both programs)

Customer	Cabinet size	Destination	Best assignment
Central Stores	W 3030	Boston	P-A
	W 3030	St Louis	S-C
Ward Gomery	W 1530	Pittsburgh	P-A
	W 3030	Pittsburgh	P-A
	W 3630	Dallas	S-E
	W 3630	Pittsburgh	P-A
	W 3630	New Orleans	S-E
Apex Land	W 1530	St Petersburg	P-A
	W 3030	St. Petersburg	P-A
	W 3630	Birmingham	S-E
Western Mail	W 1830	Denver	S-C
	W 3630	Denver	S-E
	W 3630	Los Angeles	S-E
Jensen Company	W 3030	Detroit	P-A
	W 3630	Minneapolis	S-E
	W 3630	Fort Wayne	S-E
Eastern Mail	W 3030	Bridgeport	P-A
	W 3030	Baltimore	P-A
	W 2418	Baltimore	P-A
	W 3018	Trenton	P-B
	W 2418	Bridgeport	P-A
Stacey Construction	W 2418	Philadelphia	P-A

4. The best location for a new plant in terms of raw-material supply and destination of finished products

As in the actual problem, the solution to the Walcab Company problem shows that LP methods do provide information which enables management to plan more effective use of existing resources and permits variations from present practice to be evaluated as a guide to future planning.

CHAPTER 11

Obtaining the Most Profitable Share of Your Market

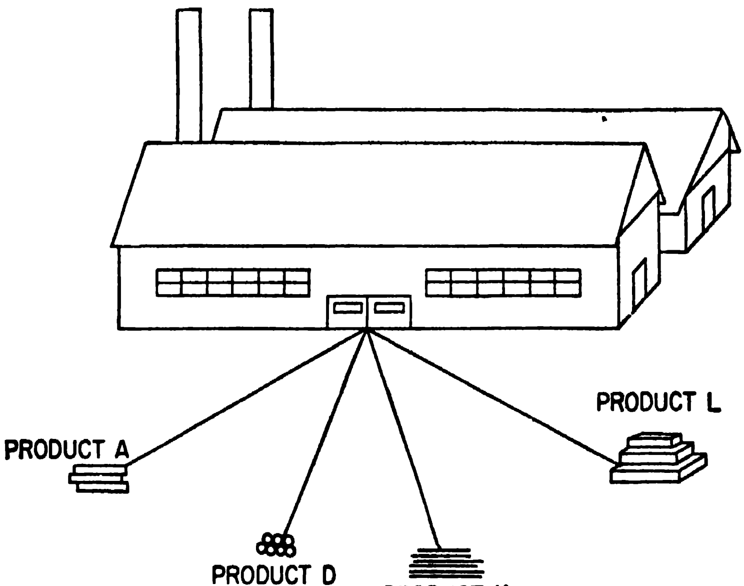


FIG. 11-1. The Textile Mills Company

Profit planning, the systematic analysis of markets, products, and facilities, is an important function of top management. The profitability and well-being of a firm is considerably influenced by how effectively the planning is done and executed. Effective planning, of course, is dependent upon accurate, up-to-date information.

Linear programming is one method available to top management for developing and evaluating information about various programs of profit under existing, as well as contemplated, conditions. In addition, the by-product information provided by the simplex method of solution makes it possible for management to evaluate the profitability of direct-

ing sales effort, adding products to the line, and increasing facilities under given sets of conditions.

This was demonstrated dramatically in the case of the Textile Mills Company, a textile firm located in the Southeastern part of the country. Because of severe competition and small profit margin, the president of the firm was vitally concerned with directing sales effort and productive capacity so as to obtain as profitable a share of the market as possible. There were differences of opinion between sales and manufacturing executives about the best course of action to be followed under existing commercial and operating policies to obtain their share of the market. Attempts on the part of the president to resolve the differences were complicated by a number of issues and problems. Part of the difficulty arose because sales and manufacturing executives looked at the problem of matching sales demands and plant capacity from different points of view. The sales executive, for example, measured plant facilities and man power primarily in terms of satisfying the customer with price, quality, and shipping date equal to or better than a competitor could provide. Where possible, of course, he had his sales force pushing the products with the highest margin per yard. From the sales point of view, the sales executive found it highly desirable to have a very flexible if not a completely elastic plant capacity.

The manufacturing executive, on the other hand, looked at the demands placed on facilities by sales in terms of the effect of those demands on balanced production, utilization of equipment and man power, and other factors that contribute to an efficient operation. The manufacturing executive championed the point of view that the desirability of satisfying varying and sometimes unexpected demands of customers must be weighed against the additional costs arising from the interruptions and changes that they bring to an even flow of work through the plant.

There was no question that the heads of both sales and manufacturing were vitally interested in lower production costs, increased sales, and higher profits. The nature of their responsibilities, however, forced them to move the firm toward these common goals by pulling in two apparently different directions.

The president, therefore, desired information that would enable him to resolve the two points of view without becoming involved in personalities. He recognized that he needed some method of evaluating the combination of sales demand and plant efficiency in terms of over-all benefit to the firm. The worth of customers' orders and tie-in sales and the profit-making potential of present and contemplated equipment had to be evaluated in the process. His problem was then reduced to the following statement:

Determine the most *profitable* market for Textile Mills[1] products and develop a sales and production program for obtaining as much of that market as commercial and production limitations permit.

BACKGROUND INFORMATION AND DESCRIPTION OF THE COMPANY OPERATION

Textile Mills have a number of plants in one location for converting synthetic and natural fibers into cloth (gray goods and fabrics). The cloth in turn is processed in subsidiary and customer plants. The firm competes for the available business with other textile firms which manufacture the same or similar products.

Basically, there are three producing departments, which correspond to the three basic steps in the manufacture of the cloth. These departments are Carding, Spinning, and Weaving.

Carding, the first step in yarn manufacture, removes foreign matter and brushes the fibers and prepares them for further processing. Spinning converts the fibers into twisted yarns for weaving. Weaving is the process of forming a woven fabric or cloth on a loom. *Fabric* is a general term for a woven material, such as cloth, lace, or hosiery. Cloths that have been woven but have not been finished are called *gray goods.*

Because of the commercial policies of customers, the efforts of competition, the product mix, and the capacity of the mill, management anticipates that they can produce and sell approximately 38,000,000 yards of cloth per year. Through selective selling, Textile Mills can affect their product mix within certain limits.

BASIC ASSUMPTIONS

For purposes of the problem, let us say that the product line consists of 12 products. They are identified in the tables in this chapter by capital letters of the alphabet. For example, the first product is identified as Product A, the second product as Product B, and so on for the 12 products.

The Profit Margin per Yard, supplied by the accounting department, is the difference between the invoice price and the variable manufacturing costs. Sales expense, general and administration expense, and such expenses as insurance, taxes, and supervision are not included in the expenses used in computing profit margins. The profit margin is a figure

[1] To preserve the confidence of the firm involved, the firm name, the product brand name, the product capacities, and the profit margins have been disguised. The problem and situation, however, are real.

that shows the margin contribution of the sales of a yard of material toward absorbing these expenses and earning profits.

An industry-wide market forecast for the coming year expressed in yards has been prepared and reflects the expected demand for the industry. The forecast indicates an industry-wide market potential of 109,500,000 yards for the year.

Departmental capacities have been adjusted for utilization. Rates of production, supplied by the Industrial-engineering Department, and profit margins, furnished by the Accounting Department, are considered fixed for the time period. This information is shown in Tables 11-1 and 11-2.

Table 11-1. Basic Product, Production Data, and Forecast Information

Product	Profit margin per yard (in dollars)	Industry-wide market forecast (in yards)	Department		
			Carding (in yards per pound)	Spinning (in yards per spindle week)	Weaving (in yards per loom week)
A	.690	2,500,000	1.35	960	15.9
B	.655	4,000,000	2.25	850	13.5
C	.610	10,000,000	1.35	960	13.4
D	.595	2,000,000	1.60	750	20.0
E	.560	4,000,000	1.70	750	18.6
F	.527	12,000,000	1.55	375	8.5
G	.491	8,000,000	1.50	655	15.5
H	.488	5,000,000	1.60	350	9.7
I	.457	6,500,000	1.23	582	13.4
J	.436	10,500,000	1.50	650	15.0
K	.411	15,000,000	1.55	605	15.1
L	.378	30,000,000	1.96	720	20.4

Table 11-2. Market Forecast and Available Capacity for the Year

Market forecast for the industry	109,500,000 yd
Average yield of Carding Department	26,400,000 lb
Production capacity of Spinning Department	50,000 spindle weeks
Production capacity of Weaving Department	2,900,000 loom weeks

The information listed in Table 11-1 has been gathered by product. Using Product A as an example and reading across the table, we see that

the profit margin at the plant is $.690 per yard and the total industry-wide demand for that particular fabric has been forecast as 2,500,000 yards. Continuing across the Product A row in the table, we see that we obtain 1.35 yards for each pound of raw material processed in the Carding Department; that each spindle in the Spinning Department will deliver 960 yards in one week, and that each loom in the Weaving Department will produce 15.9 yards in one week.

Table 11-2 actually lists the limits to the solution by listing the total forecast for the industry for all products in the problem as 109,500,000 yards per year and the annual capacity of each department based on past experience.

STATEMENT OF THE PROBLEM

The basic problem is to determine the most profitable market for the Textile Mills Company and to develop a program for obtaining as much of that market as commercial and production limitations permit.

SOLVING THE PROBLEM BY THE SIMPLEX METHOD

There are a number of LP methods available for solving this problem. The simplex method of solution will be used to find for management which products should be manufactured and in what quantities to obtain the highest margin. Further, the method will specify the greatest attainable margin, evaluate possible variations to the best program plan, and establish the steps to be taken to obtain even greater profit increases from facilities. With this information, management is in a better position to make supportable decisions affecting present and future sales and manufacturing plans.

The first part of the problem is to determine the highest-margin product mix to sell. The simplex method showed that the highest margin was not obtained simply by selling the greatest amount of highest-margin-per-yard products that the forecast demand and departmental capacities permitted. This came as something of a surprise to management, because a number of decisions—especially in competitive-bidding situations—were based on that premise.

Table 11-3 shows the program and profit margin that will result from using facilities to produce the forecast demand in order of decreasing profit margin per yard, beginning with the highest margin per yard. Under this program it is possible to manufacture the forecast demand for Products A, B, C, D, and E without exceeding the capacity of any of the producing departments. Approximately 9,100,000 yards of the 12,-000,000 yards of Product F can be produced before the capacity of the Spinning Department is exceeded, thus limiting production to one-half

Table 11-3. Program of Highest Profit Margin per Yard by Product for the Year

Product	Margin per yard (in dollars)	Annual industry-wide forecast (in million yards)	Annual production (in million yards)	Annual capacity utilization: Carding (in pounds)	Spinning (in spindle weeks)	Weaving (in loom weeks)	Profit margin (in millions of dollars)
A	.690	2.5	2.5	1,852,500	2,600	157,250	1.725
B	.655	4.0	4.0	1,776,000	4,720	296,400	2.620
C	.610	10.0	10.0	7,410,000	10,400	746,000	6.100
D	.595	2 0	2.0	1,250,000	2,660	100,000	1.190
E	.560	4.0	4.0	2,352,000	5,320	215,200	2.240
F	.527	12.0	9.1	5,870,010	24,300	1,070,280	4.796
G	491	8.0	0				
H	.488	5.0	0				
I	.457	6.5	0				
J	.436	10.5	0				
K	.411	15 0	0				
L	.378	30 0	0				
Total			31.6	20,510,510	50,000	2,585,130	18 671

the product line. On this basis it is possible to produce 31,600,000 yards at a profit margin of $18,671,000.

Very few firms are in a position to produce only those products having the largest profit margin per product. The idea is introduced here, however, for three reasons: (1) to provide a yardstick or bench mark against which various programs of production and profit can be compared, (2) to demonstrate that sales and production decisions based on highest profit margin per product do not necessarily result in the maximum profit margin for the firm, and (3) to aid in showing how LP can indicate whether it is possible to do better and, if so, how much better.

1. Setting up the problem for solution

The simplex method, like other methods of solution, formal and informal, requires a clear statement of the problem to be solved. In the case of the Textile Mills Company the first problem to be solved is to determine the highest-profit-margin program within the forecast and capacity conditions.

The objective equation

By use of the simplex approach and notation for setting up the problem for solution (given in detail in Chapter 4) the highest profit margin is expressed as follows:

Let Z = highest profit margin in dollars for the period
P = *amount* of each product to be manufactured in the highest-profit-margin program. The amount of each product that will yield the highest profit margin is to be determined
P_1 = amount of Product A to be produced in the highest-profit-margin program
P_2 = amount of Product B
P_3 = amount of Product C
P_4 = amount of Product D
P_5 = amount of Product E
P_6 = amount of Product F
P_7 = amount of Product G
P_8 = amount of Product H
P_9 = amount of Product I
P_{10} = amount of Product J
P_{11} = amount of Product K
P_{12} = amount of Product L

Any P can have a value of zero or a positive value up to the amount of the forecast demand. The simplex method is used to find the amounts of the various Ps that will permit the maximum profit margin to be obtained under the given conditions.

The capacity limitations

Departmental capacity and forecast demand limit or restrict the amount of potential profit. Expressing departmental capacity as a restriction involves either the yield or the rate of production for each product to be processed and the hours available for production of those products.

The Carding Department has an average capacity of 26,400,000 pounds. Depending upon the product, a pound of raw material will yield a different number of product yards. For example, .741 pounds of material will yield 1 yard of Product A. On the other hand, .510 pounds of material will yield 1 yard of Product L. The point is not so much the fact that different products require different amounts of raw material as the fact that LP requires the expression of data in usable units of measure. Table 11-1 shows how production data were expressed for use in the plant. Table 11-4 shows how those data were converted for analysis and

Table 11-4. Production Data Expressed in LP Units of Measure (Reciprocals of data in Table 11-1)

Product	Carding (in pounds per yard)	Spinning (in spindle weeks per yard)	Weaving (in loom weeks per yard)
A	.741	.00104	.0629
B	.444	.00118	.0741
C	.741	.00104	.0746
D	.625	.00133	.0500
E	.588	.00133	.0538
F	.645	.00267	.1176
G	.667	.00153	0645
H	.625	.00286	.1039
I	.813	.00172	.0746
J	.667	.00153	.0667
K	.645	.00165	.0662
L	.510	.00139	.0490

solution by the simplex method. This change in basic units was necessitated by the fact that departmental capacity was expressed in pounds. Consequently, demand for the product to be run (expressed in yards) when multiplied by the yield (expressed in yards per pound) must result in pounds, otherwise the data were not usable. This may be more clearly seen if we express the correct relationship in formula fashion, as follows:

$$\text{Carding Department load} = \left(\begin{matrix}\text{yield for}\\ \text{Product A}\end{matrix} \times \begin{matrix}\text{demand for}\\ \text{Product A}\end{matrix}\right) + \cdots + \left(\begin{matrix}\text{yield for}\\ \text{Product L}\end{matrix} \times \begin{matrix}\text{demand for}\\ \text{Product L}\end{matrix}\right)$$

$$\text{Pounds} = \left(\frac{\text{pounds}}{\text{yard}} \times \text{yards}\right) + \cdots + \left(\frac{\text{pounds}}{\text{yard}} \times \text{yards}\right)$$

$$\text{Pounds} = \text{pounds} + \cdots + \text{pounds}$$

Similarly, and for the same basic reason, the rates of production in the Spinning Department and the Weaving Department were converted as shown in Table 11-4. This situation is typical of the way in which data are expressed in many firms and emphasizes the importance of examining data carefully to make certain that they are expressed in the convenient unit of measure for use by linear-programming methods. On

the basis of experience to date, the conversion of data into best units of measure requires more care than any other phase of LP analysis and solution.

Departmental capacities expressed in terms of the unknowns and the proper units are as follows:

$$26{,}400{,}000 \geqq .741P_1 + .444P_2 + .741P_3 + .625P_4 + .588P_5 + .645P_6 + .667P_7 + .625P_8 + .813P_9 + .667P_{10} + .645P_{11} + .510P_{12}$$

This expression, translated into words, states that the sum of the yields times the amount of production for all products that can be run in the Carding Department is equal to or less than the 26,400,000 pounds' capacity that is available. Likewise, the capacities of the Spinning and Weaving Departments will be utilized as shown in the following expressions:

$$50{,}000 \geqq .00104P_1 + .00118P_2 + .00104P_3 + .00133P_4 + .00133P_5 + .00267P_6 + .00153P_7 + .00286P_8 + .00172P_9 + .00153P_{10} + .00165P_{11} + .00139P_{12}$$

$$2{,}900{,}000 \geqq .0629P_1 + .0741P_2 + .0746P_3 + .0500P_4 + .0538P_5 + .1176P_6 + .0645P_7 + .1039P_8 + .0746P_9 + .0667P_{10} + .0662P_{11} + 0490P_{12}$$

The market restrictions

The forecast for each product represents the anticipated maximum industry-wide demand for that product in the forthcoming year. In this problem, management wants to make the amount of each product up to and including the forecast amount that will provide the highest margin.

The market restrictions for the various products under forecast conditions expressed mathematically are as follows.

$$P_1 \leqq 2{,}500{,}000 \text{ yards}$$

(the amount of Product A (P_1) can be equal to or less than the 2,500,000 yards forecast). Similarly,

$P_2 \leqq 4{,}000{,}000$	$P_6 \leqq 12{,}000{,}000$	$P_{10} \leqq 10{,}500{,}000$
$P_3 \leqq 10{,}000{,}000$	$P_7 \leqq 8{,}000{,}000$	$P_{11} \leqq 15{,}000{,}000$
$P_4 \leqq 2{,}000{,}000$	$P_8 \leqq 5{,}000{,}000$	$P_{12} \leqq 30{,}000{,}000$
$P_5 \leqq 4{,}000{,}000$	$P_9 \leqq 6{,}500{,}000$	

2. Calculating the highest profit margin for the forecast

The computations can be simplified by multiplying the capacity and market restrictions by 1,000, thus eliminating many of the decimal points

and zeros. The first iteration is set up in matrix form by converting all inequalities to equivalent equalities, arranging the coefficients in proper sequence in columnar array, and computing the base-row values. Table 11-5 in pocket at back of the book shows the completed Iteration 1 (Program 1) and the eight subsequent iterations calculated in arriving at the highest-margin program.

The highest-margin solution and program

The final program (Program 9) shown in Table 11-5 represents the desired or ideal sales program. It is not a summary of orders received but rather the first draft of a selective-sales program that top management can review and discuss as a basis for planning to obtain greater total profit margin.

From Table 11-10 it can be seen that Textile Mills Company should sell and make Products A, B, C, D, and E to the forecast limit under both programs. Under the highest-profit-margin program, however, management must bypass 9,100,000 yards of Product F, having a margin of $.527, per yard, in favor of 8,000,000 yards of Product G, having a margin of $.491 per yard, and 7,883,000 yards of Product J, having a margin of $.436 per yard, in order to increase profits. By electing to follow and, if possible, by actually following the highest-profit-margin program, management can increase the profit-margin potential from $18,671,000 to $21,240,000, an increase of $2,569,000 from a selective-selling program that uses the same facilities. The Spinning Department limits the amount of product and profit margin that can be obtained under the given conditions because the 50,000 spindle weeks are completely utilized. There are 1,166,000 pounds of unused carding capacity and 343,000 weeks of unused loom capacity. This open capacity could be profitably used if there were more spinning capacity.

Under the existing product mix, forecast demand, rates of production, and profit margin, it is worth $284.97 in increased profit margin for each spindle week of capacity that can be added to the 50,000 spindle weeks available. This value is shown in the Base Row of Column P_{14} of Table 11-5, Program 9, and it is indicative of the type of by-product information that the simplex calculations provide (the significance of marginal values provided by the simplex method is discussed in Chapter 5). The marginal value of $284.97 per additional spindle week of capacity will continue until the capacity of another department becomes a restriction, the operation comes into complete balance, or another product comes into the program. The gain that can be expected from increasing spinning capacity can be compared with the cost of acquiring that capacity and a decision reached on the desirability of investing in additional facilities.

The desired program represents a standard or ideal that is desirable

but may not be attainable. In other words, it is one measure of the profit-earning capacity of existing facilities. Actually, the desired program represents a target that management can use for control purposes and as a standard against which to measure other planned and actual programs. Differences, if any, in these programs represent a measure of the profit margin and production that is forgone because of restrictions in sales or demand. The desired program is a guide in directing the efforts of the entire organization toward the most profitable portion of the market. This guide indicates how to coordinate selective selling, production planning and control, and methods improvements so as to obtain the most profitable portion of the market.

OTHER SALES PROGRAMS FOR MANAGEMENT CONSIDERATION

When a desired sales program has been established, the next step is to develop a sales goal in terms of actual conditions of the industry and habits of customers. For example, the Sales Department may point out that only 15 of the total number of customers purchase Products A, B, and C. Furthermore, it may not be realistic to expect to obtain the 16,500,000 yards involved because they are traditionally split among several mills by the purchasing firms who make it a practice to have several sources of supply. It may, for example, be possible to obtain only 6,000,000 yards of the total. And, to do this, it may be necessary to produce 9,000,000 yards of either Product J or Product F. We know from the simplex analysis that sales effort should be directed toward obtaining the entire amount of Product J rather than Product F, despite the fact that Product F has a profit margin per yard of $.527 compared to a profit margin of $.436 for Product J.

Similarly, we may find that a particular group of customers represent 80 per cent of the potential use for Products D and E. Here again for commercial reasons it may be possible to obtain only 2,000,000 yards and 4,000,000 yards of either Product G or Product H. Again, sales effort should be directed toward Product G rather than Product H.

Linear programming can be used to develop revisions of sales programs so that as large a profit margin as possible can be obtained under different commercial conditions. For example, management knows that the Textile Mills Company cannot sell the entire amount of the products listed in the desired or ideal sales program. If, however, the Sales Department forecasts the amounts of those products that they believe they are capable of selling, then linear-programming methods can be used to indicate the other products that should receive concentrated selling effort so as to maintain the profit margin as high as possible.

1. Revised sales program 1

With the decision made to produce the amounts of Products A, B, C, D, E, G, and J, listed in Part A of Table 11-7, the problem now is how to use the remaining capacity to obtain as much profit margin as possible. This can be done by deducting the capacity required to produce the listed amounts of these products and allocating the remaining capacity among the other products at highest profit margin, using the simplex method.

The new problem, when set up in matrix form, is given in Table 11-6. The answer to the problem shows that the highest profit margin is obtained by using the remaining capacity to manufacture 20,035,971 yards of Product L, *which has the lowest profit margin per yard of any product in the line.* The resulting profit increase in profit margin will be $7,573,597. The selection of any other product in the line will bring a smaller profit-margin increase despite the fact that the profit margins of all other products is higher.

By using the remaining capacity to manufacture Product L, the revised sales program will return a total profit margin of $17,178,597 on a total production of 37,040,000 yards. The limiting factor on production and profit is again the Spinning Department. The information about Sales Revision 1 is summarized in Sections A and B of Table 11-7.

2. Revised sales program 2

If management believes that it is possible to obtain only 15,000,000 yards of Product L, then capacity will be available for the manufacture of yet other products. The products which will add the largest amount to the profit margin can be determined again by going through the same computational process with revised data.

Using Table 11-7 as a base, we can allocate the remaining capacity among other products so as to obtain the largest addition to profit margin. When the revised conditions have been included, the problem so set up for solution by the simplex method is given in Table 11-8.

The answer given in Part B of Table 11-9 shows that the greatest increase in profit margin results when the unused capacity is devoted to producing 4,070,000 yards of Product I. The resulting profit margin from Product I totals $1,859,990, which when added to the previous program profit of $15,275,000 gives a profit margin for the revised program of $17,134,990.

Table 11-10 shows a comparison of a number of sales programs that management can consider in aligning production facilities to obtain as large a margin of profit as possible.

The over-all conclusion to be drawn from the table is that profit margin

Table 11-6. Solution and Program for Sales Revision 1

Profit margin per thousand yards (in dollars)		P_0	0 P_{13}	0 P_{14}	0 P_{15}	0 P_{21}	0 P_{23}	0 P_{24}	0 P_{26}	0 P_{27}	527 P_6	488 P_8	457 P_9	411 P_{11}	378 P_{12}	
0	P_{13}	15,612	1								645	625	813	645	510	
0	P_{14}	27.85		1							2.67	2.86	1.72	1.65	1.39	
0	P_{15}	1,823			1						117.6	103.9	74.6	66.2	49.0	
0	P_{21}	12.0				1					1					First Program
0	P_{23}	5.0					1					1				
0	P_{24}	6.5						1					1			
0	P_{26}	15.0							1					1		
0	P_{27}	30.0								1					1	
		0	0	0	0	0	0	0	0	0	−527	−488	−457	−411	−378	
0	P_{13}	5,394	1								−334.7	−423.6	182.1	39.6	0	
→ 378	P_{12}	20.04	0	7194	0	0	0	0	0	0	1.921	2.056	1.237	1.187	1	
0	P_{15}	841			1						23.47	3.16	13.99	8.04	0	
0	P_{21}	12.0				1					1				0	
0	P_{23}	5.0					1					1			0	Best Program
0	P_{24}	6.5						1					1		0	
0	P_{26}	15.0							1					1	0	
0	P_{27}	9.96								1	−1.921	−2.056	−1.237	−1.187	0	
Highest margin (in million dollars)		7.574	0	271.94	0	0	0	0	0	0	199.14	289.17	10.59	37.69	0	

Table 11-7. Revised Sales Program 1

Product	Revised amount (in million yards)	Capacity utilization			Profit (in dollars)	
		Carding (in pounds)	Spinning (in spinning weeks)	Weaving (in loom weeks)		
A	1.0	741,000	1,040	62,900	690,000	A
B	2.0	888,000	2,360	148,200	1,310,000	
C	3.0	2,223,000	3,120	223,800	1,830,000	
D	2.0	1,250,000	2,660	100,000	1,190,000	
E	4.0	2,352,000	5,320	215,200	2,240,000	
G	3.0	2,000,000	4,590	193,500	1,473,000	
J	2.0	1,334,000	3,060	133,400	872,000	
Total	17.0	10,788,000	22,150	1,077,000	9,605,000	
Unused capacity		15,612,000	27,850	1,823,000		
						B
Product L	20.04	10,218,345	27,850	981,763	7,573,597	
Total	37.04	21,006,345	50,000	2,058,763	17,178,597	

Table 11-8. Solution and Program for Sales Revision 2

			0	0	0	0	0	0	0	527	488	457	411
		P_0	P_{13}	P_{14}	P_{15}	P_{21}	P_{23}	P_{24}	P_{26}	P_6	P_8	P_9	P_{11}
0	P_{13}	7,962	1							645	625	813	645
0	P_{14}	7		1						2 67	2.86	1.72	1.65
0	P_{15}	1,150			1					117.6	103.9	74.6	66.2
0	P_{21}	12.0				1				1			
0	P_{23}	5 0					1				1		
0	P_{24}	6.5						1				1	
0	P_{26}	15.0							1				1
		0	0	0	0	0	0	0	0	−527	−488	−457	− 411
0	P_{13}	7,631	1							−616.8	−727.0	0	−134.91
→ 457	P_9	.407	0	.5814	0	0	0	0	0	1.552	1.663	1	.9593
0	P_{15}	1,120			1					1.82	−20.16	0	−5.36
0	P_{21}	12 0				1				1		0	
0	P_{23}	5 0					1				1	0	
0	P_{24}	6.5						1				0	
0	P_{26}	14.6							1			0	−.9593
Highest margin (in million dollars)		1 860	0	265.70	0	0	0	0	0	182.26	271.99	0	27.40

Table 11-9. Revised Sales Program 2

Product	Revised amount (in million yards)	Capacity utilization			Profit (in dollars)
		Carding (in pounds)	Spinning (in spinning weeks)	Weaving (in loom weeks)	
A	1.0	741,000	1,040	62,900	690,000
B	2.0	888,000	2,360	148,200	1,310,000
C	3.0	2,223,000	3,120	223,800	1,830,000
D	2.0	1,250,000	2,660	100,000	1,190,000
E	4.0	2,352,000	5,320	215,200	2,240,000
G	3.0	2,000,000	4,590	193,500	1,473,000
J	2.0	1,334,000	3,060	133,400	872,000
L	15.0	7,650,000	20,850	735,000	5,670,000
Total	32.0	18,438,000	43,000	1,749,100	15,275,000
Unused capacity		7,962,000	7,000	1,150,900	
I	4.07	330,891	7,000	30,362	1,859,990
Total	36.07	18,768,891	50,000	1,779,462	17,134,990

per yard is not the best guide to planning sales campaigns that utilize productive capacity most effectively.

A comparison of the programs for highest profit margin per yard and for the highest profit margin brings out a rather startling profit-margin difference of $2,570,000, which is a 13 per cent increase in profit margin from the same facilities—by a selective-selling campaign.

Comparing Revised Sales Program 1 with Revised Sales Program 2, we see that we can obtain virtually the same profit margin from either program. Under program 2, however, that profit margin is obtained from one million yards less than the yards produced under program 1. With this type of information there should be little doubt about the desirability of program 2.

By analyzing sales situations in this manner it is possible to establish sales and manufacturing objectives that are realistic. We know that the desired program is not likely to be obtained, but certain specific information can be developed to guide management thinking as follows:

Table 11-10. Comparison of Alternative Sales Programs
(In millions)

Product	Industry forecast	In order of highest margin per yard		Highest profit margin		Revised Program 1		Revised Program 2	
		Quantity	Profit	Quantity	Profit	Quantity	Profit	Quantity	Profit
A	$ 2.5	2.5	$ 1.725	2.5	$ 1.725	1.0	$ .690	1.0	$ .690
B	4.0	4.0	2.620	4.0	2.620	2.0	1.310	2.0	1.310
C	10.0	10.0	6.100	10.0	6.100	3.0	1.830	3.0	1.830
D	2.0	2.0	1.190	2.0	1.190	2.0	1.190	2.0	1.190
E	4.0	4.0	2.240	4.0	2.240	4.0	2.240	4.0	2.240
F	12.0	9.1	4.796	0	0	0	0	0	0
G	8.0	0	0	8.0	3.928	3.0	1.473	3.0	1.473
H	5.0	0	0	0	0	0	0	0	0
I	6.5	0	0	0	0	0	0	4.07	1.836
J	10.5	0	0	7.883	3.437	2.0	.872	2.0	.872
K	15.0	0	0	0	0	0	0	0	0
L	30.0	0	0	0	0	20.04	7.571	15.0	5.670
Total	$109.5	31.6	$18.671	38.383	$21.241	37.04	$17.179	36.07	$17.135

Table 11-11. Comparison of Capacity Utilization under Various Sales Programs
(In dollars)

Department	Highest margin per yard	Highest margin	Revised Program 1	Revised Program 2
Carding	20,510,510	25,234,460	21,006,345	18,768,891
Spinning	50,000	50,000	50,000	50,000
Weaving	2,585,130	2,556,650	2,058,763	1,779,462

1. A selective-sales program can be developed that will permit the highest profit margin possible under the existing commercial restrictions.

2. The Sales Department has been made aware of the interrelationship of product mix and profit margin per production period.

GENERAL STEPS FOR DEVELOPING A PROGRAM TO OBTAIN THE MOST PROFITABLE SHARE OF THE MARKET WITHIN SALES AND MANUFACTURING RESTRICTIONS

The general steps for developing company-wide programs for aligning sales and manufacturing effort with the most profitable portion of the market are given in Table 11-12.

Table 11-12. A Company-wide Program for Aligning Sales and Manufacturing with the Most Profitable Portion of the Market *

1. Forecast by product the total sales demand for the market served.
2. Determine the program which will produce the highest attainable margin within the limits of the forecast and plant capacity.
3. Modify this program as required to arrive at realistic sales goals.
4. Determine the plant capacity required to meet these sales requirements.
5. Compare the costs of increased capacity with the additional profit to be gained when capacity limits sales.
6. Adjust the final sales objective or increase capacity according to the cost comparison.
7. Plan engineering studies to increase production and reduce costs where desired.
8. Develop a selective-sales program to meet sales objectives.
9. Develop the sales and manufacturing procedures and reports required to sell and produce the most profitable product mix.
10. Measure the performance of sales, engineering, and production against final sales objective, planned manufacturing capacity and utilization, and desired cost reductions.
11. Correct sales, engineering, or production programs as required to maximize profits.

CHAPTER 12

Stabilizing Production and Employment Levels at Least Cost

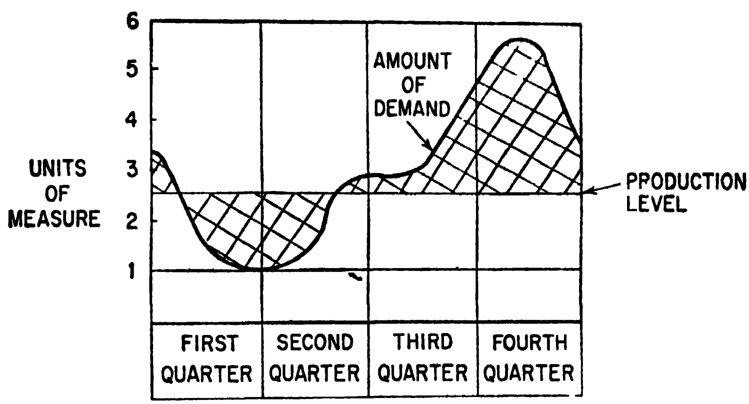


FIG. 12-1. Graph of Production and Demand Level

In the middle 1950s, stabilization of production and employment has assumed new importance—especially for firms having a seasonal or cyclical demand for their products. The issues of supplemental unemployment pay—popularly termed GAW (for "guaranteed annual wage") —are causing management to intensify efforts to find equitable solutions to meet the situation. The problem of establishing relatively stable levels of production and employment is not new. For some time, management has sought ways to level production loads that keep workers continuously employed and avoid tying up money in products that cannot be sold.

Some of the measures that have been taken to offset seasonal sales patterns include diversifying the product line, subcontracting, accumulating finished-goods inventory, offering discounts for purchases in advance of the season, hiring temporary and part-time workers, and the like. Whatever the method used to offset the effects of seasonal demand, it is strictly up to management to plan operations that solve the problem—preferably at least cost.

One of the stabilizing methods that management can put into effect quickly is the accumulation of inventory in slack periods for use and

sale in busy periods. This method has its hazards, however. It involves assuming a sales forecast for the length of the sales cycle, and it ties up company funds in merchandise held for future sales. It requires some flexibility to anticipate changes in forecast and to reestablish production plans to offset or meet the changes successfully. It also requires selecting those items to carry in inventory that will meet demand and hold carrying costs to a minimum.

Linear-programming methods have been and are being used to solve this problem at minimum inventory carrying costs. Adjustments to changes in sales forecast can be made quickly and anticipated if desired. By the use of LP methods this problem can be solved in a matter of a few hours for any desired variation from original forecast and employment levels, and management can have supportable information on which to plan.

A case in point concerns the Camtor Company (a fictitious name used to represent the firm that was involved in the real problem). The problem that management wanted solved—stated as an objective—is as follows:

> Determine the least-cost program that satisfies a fluctuating sales demand from a given amount of plant capacity and a relatively fixed level of employment, taking into account inventory accumulation costs and the option to make or buy.

The actual problem to be solved, then, is to plan a production program which will relieve the overload of the peak period and fill in the slack of the low period at least cost or penalty. The balancing is to be done by making some items ahead of the need and/or by purchasing. Figure 12-1 shows pictorially the ups and downs of the sales cycle of the firm.

From the standpoint of executive thinking, the problem was complicated by a number of conditions which had to be resolved successfully before a solution by LP methods would be acceptable. These conditions were as follows:

1. *Loss of trained personnel.* When the slow period came in the early part of each year, it was necessary to furlough some of the employees that had been hired and trained for previous peak periods. When it came time to recall the furloughed workers, few of them returned so that new employees had to be trained. This resulted in a recurring training expense which was costly because of the precision and quality requirements of the product. In addition to the preventive aspects, management was also sufficiently interested in their employees to want to provide stable employment the year round.

2. *Unused items.* No matter how carefully plans were laid to meet peak demand, it invariably happened that there was an oversupply of

some parts and a number of unfilled orders that could have been turned into sales at the peak had the right parts been available. It was expected that this would happen because it was not always possible to forecast the peak amount of sales far enough in advance to allow the lead time for making or purchasing certain parts.

3. *Fluctuating forecasts.* The Sales Department is required to forecast 6 months to 1 year in advance. This is difficult to do and yet has to be done reasonably accurately because purchases with long lead times from foreign countries are involved. The Sales Department does provide a forecast, which is revised periodically to reflect changes in demand as soon as they become known. In addition, releases are issued quarterly to manufacturing. One of the problems was how to best meet the revisions, especially as the peak period approached. Another problem was how to make a hedge in the event the forecast proved to be in error.

4. *When to start.* Since there was a question of lead times and since there was some uncertainty about how many of each part would be needed to meet peak sales, it was difficult to decide when production lots should be started in anticipation of the peak load. For several years, management had the feeling after the peak had passed that a better job of meeting peak demand could have been done had they started earlier. The next year, however, it was again a problem to decide just when to start and not incur additional inventory costs or end up with unused parts.

5. *Purchase or make.* Management had the option to purchase or make approximately one-half of the parts that went into the product line. In some cases there was a profit advantage to purchasing; in others the profit advantage came from making. Best use of the option was not being made because the uncertainty of peak requirements made it difficult to plan the use of machines, which in turn influenced purchases.

These were some of the problems that management was attempting to resolve in formulating a production, purchasing, and personnel plan that would meet company profit objectives successfully.

These problems are shown and solved successfully in the Camtor Company problem which follows. The problem presented here is quite similar to the actual problem, but scaled down for practical presentation.

The problem and its solution are presented in several parts. The first part presents the background information about the company, such as its products, inventory carrying costs, profit advantage to purchase, and the like. This information sets the stage for a restatement of the problem and the discussion of the answer.

The second part presents Program 1 and assignment of orders together

with the minimum-cost answer. The total cost or penalty of accumulating inventory and the program to be followed to obtain minimum cost are also given. In addition, the second part contains a discussion of the answer and programs and possible variations to them.

BACKGROUND INFORMATION AND DESCRIPTION OF THE PROBLEM

The Camtor Company manufactures a quality line of cameras and projectors in the middle-price range.

All the cameras and projectors are produced in one plant and distributed to dealers and retail outlets on a national basis.

The sales of cameras and projectors follow a seasonal pattern that peaks in late October, November, and early December, reflecting the Christmas trade. The low part of the cycle occurs during January and February, with a gradual upswing in the cycle starting with late March and continuing through April and May to meet the vacation demand. After that there is an accelerated increase until the October-to-December peak is reached again.

An investigation shows the same general pattern of fluctuating demand for all models. The exact pattern of demand is different for different models, however.

Manufacturing to satisfy the seasonal market pattern causes variations in the number of people required. This creates some difficulty in the case of skilled people who do not always return when called back after being furloughed.

Of the parts that go into the various models, some are manufactured, others are purchased, and a number of them can be made or purchased. The company makes it a practice to use the make-or-buy option to level employment where possible and to meet or offset the effects of unexpected changes in demand.

Production releases are issued quarterly by the Sales Department to the Manufacturing Division. Each release authorizes production of about 3 months' supply of cameras and projectors. The Production and Inventory Control Department uses the releases as the basis for planning.

The Production and Inventory Control Department specifies which parts are to be made and which are to be purchased, orders material, schedules and loads machines, and plans and follows through on the manufacture of the required number of completed models. Responsibility for the finished-goods inventory is assigned to the Sales Department.

Inventory is accumulated in anticipation of the peak demand. Inventory accumulation also helps to level production and maintain a reasonable level of employment.

Management makes every effort to provide a stable level of employment for its people. When necessary it has been the practice to work overtime and purchase certain parts rather than to hire more people during peak periods and furlough them during slow periods.

For those parts that can be either made or purchased, information is available which indicates the relative Profit Advantage (PA) of making the parts as opposed to purchasing them.

An inventory study has been made which has provided information relating to the cost of carrying inventory, the economic lot quantity to purchase, and economic amounts to hold in reserve.

A breakdown of the information necessary to solve the problem follows.

1. Parts list

The parts which have a purchase-make option and which must be manufactured are shown in Table 12-1. This table also shows the profit advantage (PA), the product on which the part is used, production hours for producing the parts for a unit of product, and the cost of carrying inventory. A discussion of the information contained in the columns of Table 12-1 is as follows:

Products and parts

In this problem only two products—a camera and a projector—and the 30 parts used in their assembly are considered to facilitate presentation. Some of the parts are used on both the camera (C) and the projector (P). These are indicated by the entries in the second column of the table.

Hours per unit

The values listed in the Hours per Unit Column represent the number of hours required to produce sufficient parts for a unit of production. In order to work with less cumbersome figures, an arbitrary unit has been set up. In the case of cameras, 1 unit is equal to 100 cameras, and for projectors, 1 unit is equal to 50 projectors. The hours required are expressed in terms of one of these units. For example, in the case of Part 6 used on cameras, the entry in the column is 80. This means that it requires 80 hours to produce the amount of Part 6 parts needed for 100 cameras. The 80 hours are arrived at as follows:

1. There are 4 of Part 6 used in each camera.
2. It takes .2 hour to produce each part.
3. It takes $4 \times .2 = .8$ hour to produce the amount of Part 6 needed for each camera.
4. On a 100-camera-unit basis, the time per unit is $.8 \times 100 = 80$ hours.

Table 12-1. Parts-list Information

Part no.	Product	Hours per unit		PA per hour (in dollars)	CCII per quarter for 1 hr production (in dollars)
		C	P		
1	C	100		20.00	1.25
2	P		50	15.00	1.46
3	C	20		10.00	.833
4	P		80	7.75	3.12
5	P		20	6.50	.417
6	C	80		6.50	.104
7	C	15		6.00	.417
8	CP	50	30	6.00	.146
9	P		10	5.75	.104
10	C	15		5.00	.125
11	C	30		4.50	.083
12	P		10	4.25	.833
13	CP	20	20	3.75	.017
14	P		20	3.50	.073
15	C	20		2.00	.625
16	C	10		1.75	.417
17	P		10	.75	.250
18	P		20	.00	.042
19	P		10	−.75	.104
20	CP	20	10	−1.00	.012
21	P		10	−3.00	.059
22	C	40		−3.25	.417
23	C	20		*M*	.625
24	C	70		*M*	6.25
25	P		40	*M*	.625
26	C	70		*M*	.833
27	P		50	*M*	2.08
28	CP	10	5	*M*	.208
29	P		14	*M*	.083
30	C	20		*M*	.083

Totals: 610 hr per unit of C
409 hr per unit of P

Profit advantage

The Profit Advantage (PA) represents the advantage in profit gained by making the part rather than buying it. A PA for any part is computed by calculating the difference in unit cost between manufacturing and purchasing and then multiplying this difference by the number of pieces produced on a machine in an hour. The product is expressed in dollars per hour and is the profit advantage of manufacturing the part. The appearance of *M* in the PA Column means that the part cannot be purchased and must be made (*M*).

Cost of carrying inventory investment

The values appearing in the Cost of Carrying Inventory Investment (CCII) Column in Table 12-1 represent the cost of carrying 1 hour's production in inventory for a quarter. These have been determined by a separate inventory study. Each value has been determined from the following formula:

$$\text{Standard cost per piece} \times \text{Pieces per hour}$$

$$\times \frac{1}{4 \text{ (quarters per year)}} \times \tfrac{1}{6} \text{ (carrying cost per year)}$$

$$= \text{CCII per quarter for 1 hour's production}$$

Table 12-3 gives the information used in arriving at the cost of carrying 1 hour of production of each part in inventory by quarters of the year. The cost to carry is considered to be zero when a part is made in a quarter for use in that quarter.

2. Machine-time requirements to meet demand

The number of parts required in each quarter expressed in hours of machine time are shown in Table 12-2. These figures represent the demand for machine time by quarters.

Bracketed figures mean that the part is used in both cameras and projectors.

The total overload figure of 7,994 hours *must* be purchased if sales demand is to be met.

Capacity in this problem also represents a level of man-hours or employment, since it requires a certain number of men to operate and service machines.

A flow chart of the general manufacturing process in shown in Figure 12-2.

Table 12-2. Requirements
(In hours of machine time)

	Quarter 1	Quarter 2	Quarter 3	Quarter 4	Total
Units C	5	7	9	10	31
P	6	7½	8	10	31½
Part 1	500	700	900	1,000	3,100
2	300	375	400	500	1,575
3	100	140	180	200	620
4	480	600	640	800	2,520
5	120	150	160	200	630
6	400	560	720	800	2,480
7	75	105	135	150	465
8	250 180	350 225	450 240	500 300	2,495
9	60	75	80	100	315
10	75	105	135	150	465
11	150	210	270	300	930
12	60	75	80	100	315
13	100 120	140 150	180 160	200 200	1,250
14	120	150	160	200	630
15	100	140	180	200	620
16	50	70	90	100	310
17	60	75	80	100	315
18	120	150	160	200	630
19	60	75	80	100	315
20	100 60	140 75	180 80	200 100	935
21	60	75	80	100	315
22	200	280	360	400	1,240
23	100	140	180	200	620
24	350	490	630	700	2,170
25	240	300	320	400	1,260
26	350	490	630	700	2,170
27	300	375	400	500	1,575
28	50 30	70 38	90 40	100 50	468
29	84	105	112	140	441
30	100	140	180	200	620
Total requirements	5,504	7,338	8,762	10,190	31,794
Capacity *	5,600	5,800	6,000	6,400	23,800
Overload	−96	1,538	2,762	3,790	7,994

* Capacity is the effective available capacity allowing for down time, setup, and expected production efficiency.

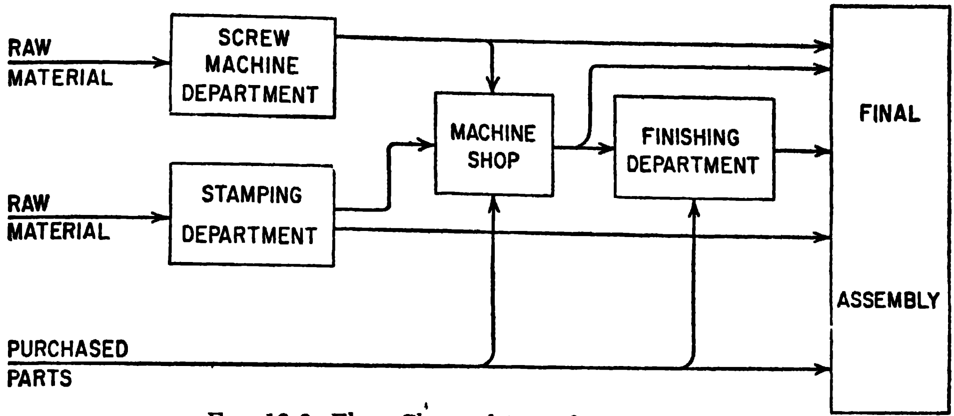


FIG. 12-2. Flow Chart of Manufacturing Process

BASIC ASSUMPTIONS FOR COMPUTATIONAL PURPOSES

In addition to the general background information, certain assumptions are made that have an important effect on the computation. This information is as follows:

1. There must be a cyclical pattern of demand, and the peak of the demand must be at the end of the period being considered.

2. Finished parts inventory—except for planned reserves—is not carried beyond the peak period. This is another way of saying that the inventory build-up that is accumulated in anticipation of the peak is assumed to be used up or exhausted by the time the peak is passed.

3. Annual demand exceeds the capacity of the plant to produce. The effect of various levels of capacity can be explored, however. For example, if capacity has been established on a two-shift basis in the past, a one-shift basis with a purchase option can be calculated to see which is the more profitable.

4. Capacity is permitted to vary some to reflect normal turnover and transfer of personnel. If desired, capacity can be treated as a constant—the same for all 4 quarters.

5. Each part can be run on only one machine. In general there are very few alternate pieces of equipment on which the various parts can be run. The alternates or choices in this problem are purchase and make-ahead in a given number of time periods.

6. All costs are based on the cost of making the item in the quarter for which it is required as zero. Costs of make-ahead or purchase are expressed as penalties or gains in relation to this zero. Since the demand exceeds the capacity, the objective is to relieve the production overload with the minimum total penalty by making ahead and purchasing parts in the proper quantities at the right time.

7. The penalty for relieving 1 hour of overload is as follows:

a. By purchasing—the penalty is equal to the profit advantage per hour. This is another way of saying that when an item is purchased on which there is a purchase-make option, an amount of profit is forgone, which is in effect a penalty.

b. By making ahead—the penalty is the cost of carrying in inventory parts that are made for a future period. Quantities of parts are expressed in terms of 1 hour's production.

8. The time period involved in 1 year is divided into quarters. The low point of the market cycle is in the first quarter. The peak is reached in the fourth quarter with a gradual build-up in between.

STATEMENT OF THE PROBLEM

The problem is as follows:

> Determine the most profitable way to meet a fluctuating sales demand from a relatively fixed amount of plant capacity (level of employment), taking into account the option to make or buy and inventory accumulation costs.

The actual problem to be solved is to find the least expensive program to level out the production load by relieving the overload of the peak period and filling in the underload of the slow period by making ahead of schedule certain items and using the purchase-make option.

SOLVING THE PROBLEM BY THE MODI METHOD

The key to setting up and solving this problem is the use of common units of measure for the penalties involved in carrying items in inventory. There is a stated penalty for making an item in a quarter before it is required and carrying it in inventory. Table 12-1 gives the penalty by part number for carrying in inventory for 1 quarter the parts made in 1 hour. For example, the value of the number of parts of Part 10 made in 1 hour is \$3.00. It costs one-sixth of the value of material, or \$.50, to carry this amount of material in the inventory 1 year. And it will cost \$.125 to carry this material in the inventory 1 quarter (of a year). This is the figure given in the right-hand column of Table 12-1. The penalty for each hour spent making Part 10 in the first quarter for use in the fourth quarter would be $(4 - 1) \times$ \$.125, or \$.38.

There is also a penalty for purchasing the amount of an item that can be made in 1 hour. For example, it is possible to make 60 pieces of Part 21 in 1 hour. Each piece costs \$.05 more when purchased than when

manufactured. In some other plant 1 hour's machine time can be purchased by buying 60 pieces of this item. The penalty or extra cost for buying this hour of machine time is 60 × \$.05, or \$3.00.

It is necessary to purchase approximately 8,000 hours' capacity in other plants. This capacity should be purchased by buying the items that have the least penalty per hour. This penalty per hour is given in the PA per Hour Column of Table 12-1. In order to level production it is also going to be necessary to make some items several months before they are needed. Here, too, the objective is to make the items ahead of demand that carry the lowest amount of penalty.

1. Setting up the problem for solution

In the actual problem the modi method of solution was used. It was chosen primarily because of the frequency with which the problem was required to be solved Further, the punched-card equipment could be used to print the problem information in modi table form, which facilitated calculations. The problem must be set up on a yearly basis to encompass a complete cycle. This is necessary if the valleys in the production cycle are to be filled in by making parts in the quarters with a low amount of demand for use in quarters with a high amount of demand.

The problem when set up for solution by the modi method is given in Table 12-3. The year has been divided into manufacturing or production periods which reflect the time of the year in which the various parts are needed. In this problem, the Camtor Sales Department issues quarterly releases to the Production-planning Department so that the year is divided into 4 quarters (the year could be divided into twelve 1-month periods or six 2-month periods). These quarters represent the periods in which parts can be made.

Since there is an option to make or purchase, purchase is shown as if it were another period of time in which parts could be made. The Purchase Hours Column in reality takes up the slack between the amount required and the amount that can be made. That amount must be purchased, otherwise delivery promises will not be kept.

The parts that are needed for use in the cameras and projectors during the year are listed in rows. Since parts will be needed or required in each quarter, each part row has four subrows—one for each quarter.

Listing the parts in a particular order in the rows will make the problem more manageable. If you return to the parts list shown in Table 12-1, you will see that some parts have a high profit advantage. It is logical to assume that these will be *made* in order to take advantage of the high PA.

***Table* 12-3. First Program and Assignment of Parts for Stabilizing Production at Least Inventory Carrying Cost**

Camtor Company

Part no.	Quarter	Col. / Row	Make hours by quarter* First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours*		Requirement hours		Total hours
23	1		100	0		−M		−M		−M		−M	100		
	2			−.63	140	0		−M		−M		−M	140		
	3			−1.25		−.63	180	0		−M		−M	180		
	4			−1.88		−1.25		−.63	200	0		−M	200		620
24	1		350	0		−M		−M		−M		−M	350		
	2			−6.25	490	0		−M		−M		−M	490		
	3			−12.50		−6.25	630	0		−M		−M	630		
	4			−18.75		−12.50		−6.25	700	0		−M	700		2,170

* —*M* indicates that the action is contrary to the conditions of the problem—it must not happen. Under Make hours by quarter for example, parts required in the first quarter cannot be assigned to a second quarter schedule. Likewise, under Purchase hours, —*M* indicates that those parts cannot be purchased; they must be made. Negative numbers are the inventory carrying costs that come from making the part before it is required.

25	1		240	0		−M		−M		−M		−M	240		
	2			−.63	300	0		−M		−M		−M	300		
	3			−1.25		−.63	320	0		−M		−M	320		
	4			−1.88		−1.25		−.63	400	0		−M	400		1,260
26	1		350	0		−M		−M		−M		−M	350		
	2			−.83	490	0		−M		−M		−M	490		
	3			−1.67		−.83	630	0		−M		−M	630		
	4			−2.50		−1 67		−.83	700	0		−M	700		2,170
27	1		300	0		−M		−M		−M		−M	300		
	2			−2.08	375	0		−M		−M		−M	375		
	3			−4.16		−2.08	400	0		−M		−M	400		
	4			−6 25		−4.16		−2 08	500	0		−M	500		1,575
28	1		80	0		−M		−M		−M		−M	80		
	2			− 21	108	0		−M		−M		−M	108		
	3			−.42		−.21	130	0		−M		−M	130		
	4			−.62		−.42		−.21	150	0		−M	150		468

Table 12-3. **First Program and Assignment of Parts for Stabilizing Production at Least Inventory Carrying Cost (*continued*)**

Camtor Company

Part no.	Quarter	Col. / Row	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours
29	1		84	0		−M		−M		−M		−M	84		
	2			−.08	105	0		−M		−M		−M	105		
	3			−.17		−.08	112	0		−M		−M	112		
	4			−.25		−.17		−.08	140	0		−M	140		441
30	1		100	0		−M		−M		−M		−M	100		
	2			−.08	140	0		−M		−M		−M	140		
	3			−.17		−.08	180	0		−M		−M	180		
	4			−.25		−.17		−.08	200	0		−M	200		.620

1	1		500	0		$-M$		$-M$		$-M$		−20.00	500		
	2			−1.25	700	0		$-M$		$-M$		−20.00	700		
	3			−2.50		−1.25	900	0		$-M$		−20.00	900		
	4			−3.75		−2.50		−1.25	1000	0		−20.00	1,000		3,100
2	1		300	0		$-M$		$-M$		$-M$		−15.00	300		
	2			−1.46	375	0		$-M$		$-M$		−15.00	375		
	3			−2.92		−1 46	400	0		$-M$		−15.00	400		
	4			−4 38		−2 92		−1.46	500	0		−15.00	500		1,575
3	1		100	0		$-M$		$-M$		$-M$		−10.00	100		
	2			−.83	140	0		$-M$		$-M$		−10.00	140		
	3			−1 67		−.83	180	0		$-M$		−10.00	180		
	4			−2.49		−1.67		−.83	200	0		−10 00	200		620
4	1		480	0		$-M$		$-M$		$-M$		−7.75	480		
	2			−3.12	600	0		$-M$		$-M$		−7 75	600		
	3			−6 24		−3.12	640	0		$-M$		−7.75	640		
	4			−9 36		−6.24		−3.12	800	0		−7.75	800		2,520

Table 12-3. **First Program and Assignment of Parts for Stabilizing Production at Least Inventory Carrying Cost (*continued*)**

Camtor Company

Part no.	Quarter	Row \ Col.	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours
5	1		120	0		$-M$		$-M$		$-M$		−6.50	120		
	2			−.42	150	0		$-M$		$-M$		−6.50	150		
	3			−.83		−.42	160	0		$-M$		−6.50	160		
	4			−1.25		−.83		−.42	200	0		−6.50	200		630
6	1		400	0		$-M$		$-M$		$-M$		−6.50	400		
	2			−.10	560	0		$-M$		$-M$		−6.50	560		
	3			−.21		−.10	720	0		$-M$		−6.50	720		
	4		90	−.31		−.21		−.10	710	0		−6.50	800		2,480

7	1		75	0		−*M*		−*M*		−*M*		−6.00	75		
	2			−.42	105	0		−*M*		−*M*		−6.00	105		
	3			−.83		−.42	135	0		−*M*		−6.00	135		
	4		150	−1.25		−.83		−.42		0		−6.00	150		465
8	1		430	0		−*M*		−*M*		−*M*		−6.00	430		
	2			−.15	575	0		−*M*		−*M*		−6.00	575		
	3		56	−.29	447	−.15	187	0		−*M*		−6.00	690		
	4		800	−.44		− 29		−.15		0		−6.00	800		2,495
9	1		60	0		−*M*		−*M*		−*M*		−5.75	60		
	2		75	−.10		0		−*M*		−*M*		−5.75	75		
	3		80	−.21		−.10		0		−*M*		−5.75	80		
	4		100	−.31		−.21		−.10		0		−5.75	100		315
10	1		75	0		−*M*		−*M*		−*M*		−5.00	75		
	2		105	−.13		0		−*M*		−*M*		−5.00	105		
	3			−.25		−.13	96	0		−*M*	39	−5.00	135		
	4			−.38		−.25		.13		0	150	−5.00	150		465

Table 12-3. First Program and Assignment of Parts for Stabilizing Production at Least Inventory Carrying Cost (*continued*)

Camtor Company

Part no.	Quarter	Col. / Row	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours
11	1			0		−*M*		−*M*		−*M*	150	−4.50	150		
	2			−.08		0		−*M*		−*M*	210	−4.50	210		
	3			−.17		−.08		0		−*M*	270	−4.50	270		
	4			−.25		−.17		−.08		0	300	−4.50	300		930
12	1			0		−*M*		−*M*		−*M*	60	−4.25	60		
	2			− 83		0		−*M*		−*M*	75	−4.25	75		
	3			−1.67		−.83		0		−*M*	80	−4.25	80		
	4			−2.50		−1.67		−.83		0	100	−4.25	100		315

13	1			0		$-M$		$-M$		$-M$	220	−3.75	220		
	2			−.02		0		$-M$		$-M$	290	−3.75	290		
	3			−.03		−.02		0		$-M$	340	−3.75	340		
	4			−.05		−.03		.02		0	400	−3.75	400		1,250
14	1			0		$-M$		$-M$		$-M$	120	−3.50	120		
	2			−.07		0		$-M$		$-M$	150	−3.50	150		
	3			−.15		−.07		0		$-M$	160	−3.50	160		
	4			−.22		−.15		−.07		0	200	−3.50	200		630
15	1			0		$-M$		$-M$		$-M$	100	−2.00	100		
	2			−.62		0		$-M$		$-M$	140	−2.00	140		
	3			−1 25		−.62		0		$-M$	180	−2.00	180		
	4			−1.88		−1.25		−.62		0	200	−2.00	200		620
16	1			0		$-M$		$-M$		$-M$	50	−1.75	50		
	2			−.42		0		$-M$		$-M$	70	−1.75	70		
	3			−.83		−.42		0		$-M$	90	−1.75	90		
	4			−1.25		−.83		−.42		0	100	−1.75	100		310

Table **12-3. First Program and Assignment of Parts for Stabilizing Production at Least Inventory Carrying Cost (*continued*)**

Camtor Company

Part no.	Quarter	Col. / Row	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours
17	1			0		−M		−M		−M	60	−.75	60		
	2			−.25	0			−M		−M	75	−.75	75		
	3			−.50		−.25	0			−M	80	−.75	80		
	4			−.75		−.50		.25		0	100	−.75	100		315
18	1		0			−M		−M		−M	120	0	120		
	2			−.04	0			−M		−M	150	0	150		
	3			−.08		−.04	0			−M	160	0	160		
	4			−.12		−.08		−.04		0	200	0	200		630

19	1			0		$-M$		$-M$		$-M$	60	+.75	60		
	2			−.10		0		$-M$		$-M$	75	+.75	75		
	3			−.21		−.10		0		$-M$	80	+.75	80		
	4			−.31		−.21		−.10		0	100	+.75	100		315
20	1			0		$-M$		$-M$		$-M$	160	+1.00	160		
	2			−.01		0		$-M$		$-M$	215	+1.00	215		
	3			−.02		−.01		0		$-M$	260	+1.00	260		
	4			−.04		−.02		−.01		0	300	+1.00	300		935
21	1			0		$-M$		$-M$		$-M$	60	+3.00	60		
	2			−.06		0		$-M$		$-M$	75	+3.00	75		
	3			−.12		−.06		0		$-M$	80	+3.00	80		
	4			−.18		−.12		−.06		0	100	+3.00	100		315
22	1			0		$-M$		$-M$		$-M$	200	+3.25	200		
	2			−.42		0		$-M$		$-M$	280	+3.25	280		
	3			−.83		−.42		0		$-M$	360	+3.25	360		
	4			−1.25		−.83		−.42		0	400	+3.25	400		1,240
Machine hours available			5,600		5,800		6,000		6,400		7,994		31,794		31,794

Other parts have a negative profit advantage, and it is logical to assume that these will probably be *purchased.*

Still other parts cannot be purchased and must be made. These are designated by the letter *M* in the PA Column, meaning "*M*ust be made—*M*ust not be purchased."

The order in which the parts are assigned to rows is as follows:

1. All parts which must be made are listed first. These include Parts 23 to 30.

2. All remaining parts in order of decreasing PA are listed as shown in Table 12-3.

The remaining information in Table 12-1 is used as follows:

1. The PAs are listed in the small boxes in the Purchase Hours Column. For example, Part 1 has a PA of \$20 irrespective of the quarter in which it is purchased. Therefore the value \$20 appears in each box of the Purchase Hours Column opposite Part 1. The PAs are given a minus sign because they represent a penalty for purchasing instead of making. The parts which had a minus PA now have a plus PA under these conditions. In effect all PAs have been multiplied by a minus one (-1).

The cost of carrying inventory investment values are listed in the cost boxes in the Make Hours Columns. Any part made in a quarter for use in that quarter has a zero CCII. For example, Part 23 made in the first quarter for the first quarter has a zero (0) in the cost box in the square corresponding to the intersection of the row and column. Part 23 made in the second quarter for the second quarter also has a zero in the cost box. On the other hand, Part 23 for the first quarter cannot be made in the second quarter because by then the first quarter would have already gone by. Consequently, the cost box is blanked out or is assigned a value of $-M$, which indicates "*M*ust not happen." Also holding a part in inventory is prohibitive by price and by the ground rules set up. Part 23 made in the first quarter for the second quarter would involve an additional penalty or cost of carrying the part in inventory for a quarter. In this case the value listed in the cost box would be —\$.625. If Part 23 were made in the first quarter for the third quarter, the value in the cost box would be $2 \times .625 = \$1.25$. The $-\$1.25$ represents the penalty or cost of carrying 1 hour's production in inventory for 6 months, or 2 quarters.

Table 12-2 contains the requirements for parts in hours by quarter. These are inserted in the appropriate place in the Requirement Hours Column of Table 12-3. Table 12-2 also contains the adjusted or effective capacity by quarter. These values are placed in the bottom row of Table 12-3 in the appropriate quarter column. The total overload values shown in Table 12-2 are the difference between the number of hours of pro-

duction needed to satisfy the demand and the number of hours of production available. The difference is the number of hours of production that needs to be *purchased* from outside suppliers. The total overload hours, or purchase value—in this case 7,994 hours—are placed at the bottom of the Purchase Hours Column of Table 12-3.

At this point, all information has been placed in the table, and the actual assigning of production to quarters can be carried out and the problem solved.

2. Calculating the program of least inventory carrying cost

There are a number of ways in which the capacity can be assigned. One way that has proved successful is to assign or load the quarters by first assigning the items to be purchased. The procedure for doing this is to start at the bottom of the Purchase Column and move up the column, accumulating purchase hours until the required number has been reached or purchased. This in effect establishes a cutoff point, and the quarters are then loaded for make requirement starting at the top and working down until the available capacity has been assigned. After all columns and rows are checked to see that requirements, capacities, and purchases have been met, the problem is then solved by the modi method as described in Chapter 3. In this problem, four problems (matrices) are being solved at one time. Table 12-4 gives the least-penalty solution and program for meeting the forecast under the conditions of the problem.

INTERPRETATION AND SIGNIFICANCE OF THE BEST PROGRAM

The best program—the one that minimizes the penalty while maintaining a relatively fixed level of capacity (employment)—is obtained by making certain items ahead of time to level production rather than going by straight purchase-make decisions. The program also indicates what items should be made ahead, in what quantities, and in which quarters to minimize inventory carrying costs.

A breakdown and analysis of the least-cost program by quarters is as follows:

1. Program for the first quarter of the year

The problem for the first quarter, shown in Table 12-5, indicates that it is necessary to produce *some items for later quarters as early as the first quarter.* In the program of least penalty, 359 hours of first-quarter capacity are assigned for production of parts for the fourth quarter. With-

Table 12-4. **Program of Least Inventory Cost to Stabilize Production**

Camtor Company

Part no.	Make quarter	Row \ Col.	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours
			0		+.10		+.21		+.31		−4.79				
23	1	0	100	0		−*M*		−*M*		−*M*		−*M*	100		
	2	−.10	−.10	−.63	140	0		−*M*		−*M*		−*M*	140		
	3	−.21	−.21	−1.25	−.11	−.63	180	0		−*M*		−*M*	180		
	4	−.31	−.31	−1.88	−.21	−1.25	−.10	−.63	200	0		−*M*	200		620
24	1	0	350	0		−*M*		−*M*		−*M*		−*M*	350		
	2	−.10	−.10	−6.25	490	0		−*M*		−*M*		−*M*	490		
	3	−.21	−.21	−12.50	−.11	−6.25	630	0		−*M*		−*M*	630		
	4	−.31	−.31	−18.75	−.21	−12.50	−.10	−6.25	700	0		−*M*	700		2,170
25	1	0	240	0		−*M*		−*M*		−*M*		−*M*	240		
	2	−.10	−.10	−.63	300	0		−*M*		−*M*		−*M*	300		
	3	−.21	−.21	−1.25	−.11	−.63	320	0		−*M*		−*M*	320		
	4	−.31	−.31	−1.88	−.21	−1.25	−.10	−.63	400	0		−*M*	400		1,260

26	1	0	350	0		−M		−M		−M		−M	350		
	2	−.10	−.10	−.83	490	0		−M		−M		−M	490		
	3	−.21	−.21	−1.67	−.11	−.83	630	0		−M		−M	630		
	4	−.31	−.31	−2.50	−.21	−1.67	−.10	−.83	700	0		−M	700		2,170
27	1	0	300	0		−M		−M		−M		−M	300		
	2	−.10	−.10	−2.08	375	0		−M		−M		−M	375		
	3	−.21	−.21	−4.16	−.11	−2.08	400	0		−M		−M	400		
	4	−.31	−.31	−6.25	−.21	−4.16	−.10	−2.08	500	0		−M	500		1,575
28	1	0	80	0		−M		−M		−M		−M	80		
	2	−.10	−.10	−.21	108	0		−M		−M		−M	108		
	3	−.21	−.21	−.42	−.11	−.21	130	0		−M		−M	130		
	4	−.31	−.31	−.62	−.21	−.42	−.10	−.21	150	0		−M	150		468
29	1	0	84	0		−M		−M		−M		−M	84		
	2	−.08	105	−.08	+.02	0		−M		−M		−M	105		
	3	−.17	112	−.17	−.07	−.08	+.04	0		−M		−M	112		
	4	−.25	140	−.25	−.15	−.17	−.04	−.08	+.06	0		−M	140		441
30	1	0	100	0		−M		−M		−M		−M	100		
	2	−.08	140	−.08	+.02	0		−M		−M		−M	140		
	3	−.17	180	−.17	−.07	.08	+.04	0		−M		−M	180		
	4	−.25	200	−.25	−.15	−.17	−.04	−.08	+.06	0		−M	200		620

Table 12-4. Program of Least Inventory Cost to Stabilize Production (*continued*)

Camtor Company

Part no.	Make quarter	Row \ Col.	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours
			0		+.10		+.21		+.31		−4.79				
1	1	0	500	0		−*M*		−*M*		−*M*	0	−20.00	500		
	2	−.10	−.10	−1.25	700	0		−*M*		−*M*	0	−20.00	700		
	3	−.21	−.21	−2.50	−.11	−1.25	900	0		−*M*	0	−20.00	900		
	4	−.31	−.31	−3.75	−.21	−2.50	−.10	−1.25	1000	0	0	−20.00	1,000		3,100
2	1	0	300	0		−*M*		−*M*		−*M*	0	−15.00	300		
	2	−.10	−.10	−1.46	375	0		−*M*		−*M*	0	−15.00	375		
	3	−.21	−.21	−2.92	−.11	−1.46	400	0		−*M*	0	−15.00	400		
	4	−.31	−.31	−4.38	−.21	−2.92	−.10	−1.46	500	0	0	−15.00	500		1,575
3	1	0	100	0		−*M*		−*M*		−*M*	0	−10.00	100		
	2	−.10	−.10	−.83	140	0		−*M*		−*M*	0	−10.00	140		
	3	−.21	−.21	−1.67	−.11	−.83	180	0		−*M*	0	−10.00	180		
	4	−.31	−.31	−2.49	−.21	1.67	−.10	−.83	200	0	0	−10.00	200		620

4	1	0	480	0		$-M$		$-M$		$-M$	0	−7.75	480		
	2	−.10	−.10	−3.12	600	0		$-M$		$-M$	0	−7.75	600		
	3	−.21	−.21	−6.24	−.11	−3.12	640	0		$-M$	0	−7.75	640		
	4	−.31	−.31	−9.36	−.21	−6.75	−.10	−3.12	800	0	0	−7.75	800		2,520
5	1	0	120	0		$-M$		$-M$		$-M$	0	−6.50	120		
	2	−.10	−.10	−.42	150	0		$-M$		$-M$	0	−6.50	150		
	3	−.21	−.21	−.83	−.11	−.42	160	0		$-M$	0	−6.50	160		
	4	−.31	−.31	−1.25	−.21	−.83	−.10	−.42	200	0	0	−6.50	200		630
6	1	0	400	0		$-M$		$-M$		$-M$	0	−6.50	400		
	2	−.10	560	−.10		0		$-M$		$-M$	0	−6.50	560		
	3	−.20	−.20	−.21	720	−.10	+.01	0		$-M$	0	−6.50	720		
	4	−.31	19	−.31	172	−.21	509	−.10	100	0	0	−6.50	800		2,480
7	1	0	75	0		$-M$		$-M$		$-M$	0	−6.00	75		
	2	−.10	−.10	−.42	105	0		$-M$		$-M$	0	−6.00	105		
	3	−.21	−.21	−.83	−.11	−.42	135	0		$-M$	0	−6.00	135		
	4	−.31	−.31	−1.25	−.21	−.83	−.10	−.42	150	0	0	−6.00	150		465
8	1	0	430	0		$-M$		$-M$		$-M$	0	−6.00	430		
	2	−.10	−.10	−.15	575	0		$-M$		$-M$	0	−6.00	575		
	3	−.21	−.21	−.29	−.11	−.15	690	0		$-M$	0	−6.00	690		
	4	−.31	−.11	−.44	−.21	−.29	−.10	−.15	800	0	0	−6.00	800		2,495

Table 12-4. Program of Least Inventory Cost to Stabilize Production (*continued*)

Camtor Company

Part no.	Make quarter	Col. / Row	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours	
			0		+.10		+.21		+.31		−4.79					
9	1	0	60	0		−*M*		−*M*		−*M*	0	−5.75	60			
	2	−.10	−.10	−.10	75	0		−*M*		−*M*	0	−5.75	75			
	3	−.20	−.20	−.21	80	−.10	+.01	0		−*M*	0	−5.75	80			*
	4	−.31	−.31	−.31	100	−.21	−.10	−.10	0	0	0	−5.75	100		315	
10	1	0	75	0		−*M*		−*M*		−*M*	0	−5.00	75			
	2	−.10		−.13	105	0		−*M*		−*M*	0	−5.00	105			
	3	−.21	−.21	−.25	−.11	−.13	96	0		−*M*	39	−5.00	135	195.00		
	4	−.21	−.21	−.38	−.21	−.25	−.10	−.13	+.10	0	150	−5.00	150	750.00	465	
11	1	+.29	+.29	0		−*M*		−*M*		−*M*	150	−4.50	150	675.00		
	2	+.29	+.29	−.08	+.39	0		−*M*		−*M*	210	−4.50	210	945.00		†
	3	+.29	+.29	−.17	+.39	−.08	+.50	0		−*M*	270	−4.50	270	1,215.00		
	4	+.29	+.29	−.25	+.39	−.17	+.50	−.08	+.60	0	300	−4.50	300	1,350.00	930	

12	1	+.54	+.54	0		$-M$		$-M$		$-M$	60	−4.25	60	255.00	
	2	+.54	+.54	−.83	+.64	0		$-M$		$-M$	75	−4.25	75	318.75	
	3	+.54	+.54	−1.67	+.64	−.83	+.75	0		$-M$	80	−4.25	80	340.00	
	4	+.54	+.54	−2.50	+.64	−1.67	+.75	−.83	+.85	0	100	−4.25	100	425.00	315
13	1	+1.04	+1.04	0		$-M$		$-M$		$-M$	220	−3.75	220	825.00	
	2	+1.04	+1.04	−.02	+1.14	0		$-M$		$-M$	290	−3.75	290	1,087.50	
	3	+1.04	+1.04	−.03	+1.14	−.02	+1.25	0		$-M$	340	−3.75	340	1,275.00	
	4	+1.04	+1.04	−.05	+1.14	−.03	+1.25	−.02	+1.35	0	400	−3.75	400	1,500.00	1,250
14	1	+1.29	+1.29	0		$-M$		$-M$		$-M$	120	−3.50	120	420.00	
	2	+1.29	+1.29	−.07	+1.39	0		$-M$		$-M$	150	−3.50	150	525.00	
	3	+1.29	+1.29	−.15	+1.39	−.07	+1.50	0		$-M$	160	−3.50	160	560.00	
	4	+1.29	+1.29	−.22	+1.39	−.15	+1.50	−.07	+1.60	0	200	−3.50	200	700.00	630
15	1	+2.79	+2.79	0		$-M$		$-M$		$-M$	100	−2.00	100	200.00	
	2	+2.79	+2.79	−.62	+2.89	0		$-M$		$-M$	140	−2.00	140	280.00	
	3	+2.79	+2.79	−1.25	+2.89	−.62	+3.00	0		$-M$	180	−2.00	180	360.00	
	4	+2.79	+2.79	−1.88	+2.89	−1.25	+3.00	−.62	+3.10	0	200	−2.00	200	400.00	620
16	1	+3.04	+3.04	0		$-M$		$-M$		$-M$	50	−1.75	50	87.50	
	2	+3.04	+3.04	−.42	+3.14	0		$-M$		$-M$	70	−1.75	70	122.50	
	3	+3.04	+3.04	−.83	+3.14	−.42	+3.25	0		$-M$	90	−1.75	90	157.50	
	4	+3.04	+3.04	−1.25	+3.14	−.83	+3.25	−.42	+3.35	0	100	−1.75	100	175.00	310

†

Table 12-4. Program of Least Inventory Cost to Stabilize Production (*continued*)

Camtor Company

Part no.	Make quarter	Col. / Row	Make hours by quarter: First quarter		Second quarter		Third quarter		Fourth quarter		Purchase hours		Requirement hours		Total hours
			0		+.10		+.21		+.31		−4.79				
17	1	+4.04	+4.04	0		−M		−M		−M	60	−.75	60	45.00	
	2	+4.04	+4.04	−.25	+4.14	0		−M		−M	75	−.75	75	56.25	
	3	+4.04	+4.04	−.50	+4.14	−.25	+4.25	0		−M	80	−.75	80	60.00	
	4	+4.04	+4.04	−.75	+4.14	−.50	+4.25	−.25	+4.35	0	100	−.75	100	75.00	315
18	1	+4.79	+4.79	0		−M		−M		−M	120	0	120	0	
	2	+4.79	+4.79	−.04	+4.89	0		−M		−M	150	0	150	0	
	3	+4.79	+4.79	−.08	+4.89	−.04	+5.00	0		−M	160	0	160	0	
	4	+4.79	+4.79	−.12	+4.89	−.08	+5.00	−.04	+5.10	0	200	0	200	0	630
19	1	+5.54	+5.54	0		−M		−M		−M	60	+.75	60	45.00	
	2	+5.54	+5.54	−.10	+5.64	0		−M		−M	75	+.75	75	56.25	
	3	+5.54	+5.54	−.21	+5.64	−.10	+5.85	0		−M	80	+.75	80	60.00	
	4	+5.54	+5.54	−.31	+5.64	−.21	+5.85	−.10	+5.95	0	100	+.75	100	75.00	315

†

20	1	+5.79	+5.79	0		$-M$		$-M$		$-M$	160	+1.00	160	160.00	
	2	+5.79	+5.79	−.01	+5.89	0		$-M$		$-M$	215	+1.00	215	215.00	
	3	+5.79	+5.79	−.02	+5.89	−.01	+6 00	0		$-M$	260	+1.00	260	260.00	
	4	+5.79	+5.79	−.04	+5.89	−.02	+6 00	− 01	+6.10	0	300	+1.00	300	300.00	935
21	1	+7.79	+7.79	0		$-M$		$-M$		$-M$	60	+3.00	60	180.00	
	2	+7.79	+7.79	−.06	+7.89	0		$-M$		$-M$	75	+3.00	75	225.00	
	3	+7.79	+7.79	−.12	+7.89	−.06	+8.00	0		$-M$	80	+3.00	80	240.00	
	4	+7.79	+7.79	−.18	+7 89	−.12	+8.00	−.06	+8.10	0	100	+3.00	100	300.00	315
22	1	+8.04	+8.04	0		$-M$		$-M$		$-M$	200	+3.25	200	650.00	
	2	+8.04	+8.04	−.42	+8 14	0		$-M$		$-M$	280	+3.25	280	910.00	
	3	+8.04	+8.04	−.83	+8.14	−.42	+8.25	0		$-M$	360	+3.25	360	1,170.00	
	4	+8.04	+8.04	−1.25	+8.14	−.83	+8.25	−.42	+8.35	0	400	+3.25	400	1,300.00	1,240
Machine hours available			5,600		5,800		6,000		6,400		7,994		31,794		31,794

Minimum penalty to stabilize production (cost to carry make-ahead inventory) = $404.15

Make-ahead hours	1,456	1,072	509	0	Total: 3,037 hours

Purchase items required to satisfy stable production program = 50 items

* Make items. Cost to carry make-ahead items $404.15.
† Purchase items. Penalty—$9,233.75.

Table 12-5. Purchase-make Program for First Quarter

Part	Make	Part	Purchase
1	500 hr for first quarter	11	150 hr for first quarter
2	300 hr for first quarter	12	60 hr for first quarter
3	100 hr for first quarter	13	220 hr for first quarter
4	480 hr for first quarter	14	120 hr for first quarter
5	120 hr for first quarter	15	100 hr for first quarter
6	400 hr for first quarter	16	50 hr for first quarter
6	560 hr for second quarter	17	60 hr for first quarter
6	19 hr for fourth quarter	18	120 hr for first quarter
7	75 hr for first quarter	19	60 hr for first quarter
8	430 hr for first quarter	20	160 hr for first quarter
9	60 hr for first quarter	21	60 hr for first quarter
10	75 hr for first quarter	22	200 hr for first quarter
23	100 hr for first quarter		
24	350 hr for first quarter		
25	240 hr for first quarter		
26	350 hr for first quarter		
27	300 hr for first quarter		
28	80 hr for first quarter		
29	84 hr for first quarter		
29	105 hr for second quarter		
29	112 hr for third quarter		
29	140 hr for fourth quarter		
30	100 hr for first quarter		
30	140 hr for second quarter		
30	180 hr for third quarter		
30	200 hr for fourth quarter		
Total	5,600 hr	Total	1,360 hr

First-quarter Hours Assigned to Make-ahead for Other Quarters

Part	Hours for second quarter	Part	Hours for third quarter	Part	Hours for fourth quarter
6	560	29	112	6	19
29	105	30	180	29	140
30	140			30	200
Total	805		292		359

Total make-ahead hours: 1,456

out the LP solution, it is not obvious that certain parts for the fourth quarter must be produced in the first 3 months of the year if the forecast is to be met at minimum cost. The number of hours of production and the particular parts that should be produced ahead of need are even less obvious. Yet the LP solution specifies them clearly and provides management with a specific time to begin production to meet fourth-quarter demand. Parts to be produced in the first quarter for the fourth quarter are Part 6, 19 hours; Part 29, 140 hours; and Part 30, 200 hours.

The first-quarter program also includes production for the third and second quarters. The third-quarter items take 292 hours—which is less than the fourth quarter. The parts to be produced for the third quarter are Part 29, 112 hours; and Part 30, 180 hours.

The second-quarter items take 805 hours. The parts to be produced for the second quarter are Part 6, 560 hours; Part 29, 105 hours; and Part 30, 140 hours.

The number of hours of production to be purchased in the first quarter totals 1,360 hours. This means that less penalty or cost is incurred by using capacity to make parts for later quarters and absorbing the increased inventory cost than by making the items that are purchased for first-quarter use.

2. Program for the second quarter of the year

Like the first-quarter program, certain parts are required to be made in the second quarter for use in the third and fourth quarters to minimize penalty, or cost, in meeting the forecast. Parts that are required to be made for the third quarter are Part 6, 720 hours; and Part 9, 80 hours. Parts to be produced in the second quarter for the fourth quarter are Part 6, 172 hours; and Part 9, 100 hours. The purchase items for the second quarter are Parts 11 to 22, totaling 1,805 hours.

3. Program for the third quarter of the year

The 509 hours of make-ahead in the third quarter are assigned to producing Part 6 requirements for the fourth quarter. The purchase items for the third quarter are again Parts 11 to 22, plus a part of the requirement of Part 10. The other portion of Part 10 requirements is made in the third quarter.

4. Program for the fourth quarter of the year

There are no make-ahead items in the fourth quarter since the fourth quarter is the last one considered under the current program. The items to be purchased are Parts 10 to 22. The total hours of production to be purchased in all 4 quarters amount to 7,994 hours. The number of hours

Table 12-6. Purchase-make Program for Second Quarter

Part	Make	Part	Purchase
1	700 hr for second quarter	11	210 hr for second quarter
2	375 hr for second quarter	12	75 hr for second quarter
3	140 hr for second quarter	13	290 hr for second quarter
4	600 hr for second quarter	14	150 hr for second quarter
5	150 hr for second quarter	15	140 hr for second quarter
6	720 hr for third quarter	16	70 hr for second quarter
6	172 hr for fourth quarter	17	75 hr for second quarter
7	105 hr for second quarter	18	150 hr for second quarter
8	575 hr for second quarter	19	75 hr for second quarter
9	75 hr for second quarter	20	215 hr for second quarter
9	80 hr for third quarter	21	75 hr for second quarter
9	100 hr for fourth quarter	22	280 hr for second quarter
10	105 hr for second quarter		
23	140 hr for second quarter		
24	490 hr for second quarter		
25	300 hr for second quarter		
26	490 hr for second quarter		
27	375 hr for second quarter		
28	108 hr for second quarter		
29	0 hr for second quarter		
30	0 hr for second quarter		
Total	5,800 hr	Total	1,805 hr

Second-quarter Hours Assigned to Make-ahead for Other Quarters

Part	Hours for third quarter	Part	Hours for fourth quarter
6	720	6	172
9	80	9	100
Total	800		272

Total make-ahead hours: 1,072

Table 12-7. Purchase-make Program for Third Quarter

Part	Make	Part	Purchase
1	900 hr for third quarter	10	39 hr for third quarter
2	400 hr for third quarter	11	270 hr for third quarter
3	180 hr for third quarter	12	80 hr for third quarter
4	640 hr for third quarter	13	340 hr for third quarter
5	160 hr for third quarter	14	160 hr for third quarter
6	509 hr for fourth quarter	15	180 hr for third quarter
7	135 hr for fourth quarter	16	90 hr for third quarter
8	690 hr for fourth quarter	17	80 hr for third quarter
9	0 hr for fourth quarter	18	160 hr for third quarter
10	96 hr for fourth quarter	19	80 hr for third quarter
		20	260 hr for third quarter
23	180 hr for fourth quarter	21	80 hr for third quarter
24	630 hr for fourth quarter	22	360 hr for third quarter
25	320 hr for fourth quarter		
26	630 hr for fourth quarter		
27	400 hr for fourth quarter		
28	130 hr for fourth quarter		
29	0 hr for fourth quarter		
30	0 hr for fourth quarter		
Total	6,000 hr	Total	2,179 hr

Third-quarter Hours Assigned to Make-ahead for Other Quarters

Part 6: 509 hr for fourth quarter

Total make-ahead: 509
There is no make-ahead in the fourth quarter.

Table 12-8. Purchase-make Program for Fourth Quarter

Part	Make	Part	Purchase
1	1,000 hr for fourth quarter	10	150 hr for fourth quarter
2	500 hr for fourth quarter	11	300 hr for fourth quarter
3	200 hr for fourth quarter	12	100 hr for fourth quarter
4	800 hr for fourth quarter	13	400 hr for fourth quarter
5	200 hr for fourth quarter	14	200 hr for fourth quarter
6	100 hr for fourth quarter	15	200 hr for fourth quarter
7	150 hr for fourth quarter	16	100 hr for fourth quarter
8	800 hr for fourth quarter	17	100 hr for fourth quarter
9	0 hr for fourth quarter	18	200 hr for fourth quarter
10	0 hr for fourth quarter	19	100 hr for fourth quarter
		20	300 hr for fourth quarter
23	200 hr for fourth quarter	21	100 hr for fourth quarter
24	700 hr for fourth quarter	22	400 hr for fourth quarter
25	400 hr for fourth quarter		
26	700 hr for fourth quarter		
27	500 hr for fourth quarter		
28	150 hr for fourth quarter		
29	0 hr for fourth quarter		
30	0 hr for fourth quarter		
Total	6,400 hr	Total	2,650 hr

purchased per quarter have doubled from 1,360 hours in the first quarter to 2,650 hours in the fourth quarter.

5. Total make ahead

A summary of the hours assigned to make and purchase by quarter is given in Table 12-9.

Table 12-9 Total Hours Purchase-make to Meet Forecast at Minimum Cost

Quarter	Total hours make	Total hours purchase
First	5,600	1,360
Second	5,800	1,805
Third	6,000	2,179
Fourth	6,400	2,650
Total	**23,800**	**7,994**

Penalty

The inventory penalty incurred by making ahead to level production is as follows:

Part 29:	105 × .08	8.40
	112 × .17	19.04
	140 × .25	35.00
Part 30:	140 × .08	11.20
	180 × .17	30.60
	200 × .25	50.00
Part 6:	560 × .10	56.00
	19 × .31	5.89
	720 × .10	72.00
	172 × .21	36.12
	509 × .10	50.90
Part 9:	80 × .10	8.00
	100 × .21	21.00
	Total	404.15

This penalty represents the additional cost of carrying inventory to meet a fairly stable level of production and employment.

At this point the program of least penalty—exercising a purchase-make option and accumulating inventory—provides the following information to management for planning purposes.

1. A definite program which specifies which items to make ahead and the amount to carry in inventory.

2. The penalty, or cost in dollars and cents, shown in Table 12-10 as Program 1, for conforming to the program. This cost can be compared to the training, clerical, scrap, and unemployment costs involved in furloughing and hiring personnel.

3. A definite time has been established at which the various parts must be started to meet forecast. This eliminates the uncertainty about when to start.

4. A program of purchasing can be worked out in advance.

5. There will be no unused parts left over at the end of the year—provided the forecast is accurate.

Now that the problem has been set up and a formalized method established for solving it, management can test and explore various alternative programs, as well as keep abreast of changes as they become known—including unplanned failures in equipment, unforeseen changes in demand, and unplanned changes in price.

ALTERNATIVE PROGRAMS

Useful information for management planning can be obtained by considering variations to the basic program.

Program of least inventory cost to stabilize production

Some of the least-inventory-cost variations that can be considered are as follows:

1. Scheduling an exactly constant load
2. Increasing the forecast 10 per cent as a hedge
3. Revising the program for the last 4 months to correct for earlier unplanned variations
4. Determining the profit-making potential of new machinery
5. Correcting for decrease in forecast—last 6 months
6. Determining the economy of overtime operation during the peak period

1. Scheduling an exactly constant load of 5,800 machine hours per quarter

In some instances management may want to know what is involved in stabilizing available machine hours at one level for the entire year. Table 12-10, Program 2, shows the variations resulting from establishing

Table 12-10. Comparison of Alternative Programs

Program	Make-ahead hours				Purchase			Inventory (make-ahead) penalty (in dollars)	Total (in dollars)
	First quarter	Second quarter	Third quarter	Fourth quarter	Items	Hours	Penalty (in dollars)		
Program 1 Small-capacity variations per quarter	1,456	1,072	509	3,037	50	7,994	9,233.75	404.15	9,637.90
Program 2 5,800 hours' capacity per quarter	1,791	1,252	405	3,448	57	8,594	12,479.00	571.13	13,050.33
Program 3 10 per cent increase in sales	1,189	498	530	2,217	59	11,293	23,977.50	294.84	24,272.34
Program 4 Revised last 4 months	10	0	0	10	57	4,434	8,612.50	.40	8,612.90

available machine hours at a constant level of 5,800 hours per quarter. Any number of machine hours can be assumed. The number 5,800 was chosen arbitrarily in this case to show the kind of analysis that can be made. For a 5,800-hour-per-quarter program, the table shows a 10 per cent increase in the number of make-ahead hours—3,037 to 3,448, despite fewer total machine hours—23,200 to 23,800. This results in an increase in the inventory carrying costs—$404.15 to $571.13—and in the number of items purchased—50 to 57. The additional items of purchase are significant because they represent items that have a high profit advantage if made. The purchase profit forgone (penalty) is $3,100 greater for the year.

The specific programs for the first quarter are almost identically the same except for Parts 6, 9, and 10. In the 5,800-hour program, 354 hours of production for the fourth quarter are made ahead compared to 19 hours. Also Parts 9 and 10 are purchased instead of made in the first quarter. In general, the programs for the remaining 3 quarters compare about the same way. The important observations to be obtained from the comparisons are as follows:

1. Comparisons can be made between programs by choosing any reasonable number of available machine hours per quarter.

2. A specific program of make-ahead and purchase, together with the penalties and costs, is provided.

3. Once a program is set up, effort need be concentrated in only a few parts, and the program indicates which ones. In the two programs considered, the same few parts come up for attention.

4. It is important to get off to a good start early in the year. Both programs show the heaviest make-ahead schedule in the first quarter and spell it out specifically. This perhaps more than any other single piece of information is most important to management planning. It is difficult to make up in the fourth quarter for steps not taken in the early part of the year. When they can be made up it usually involves considerable additional expense.

2. Increasing the forecast 10 per cent as a hedge

Both Program 1 and Program 2 indicate the importance of making ahead in the first quarter to level production and meet a peak fourth-quarter demand.

Knowing the importance of the first quarter, management may want to consider for their own thinking the program that would satisfy a 10 per cent increase in forecast as a hedge—should it develop. This can be worked out without much difficulty once the initial program is set up.

Table 12-10 gives a summary comparison of Program 1 and Program 3—the basic program to a 10 per cent increase. In general, it bears out

the importance of making ahead in the first quarter. Over all, the program to hedge on the forecast, however, has considerably fewer make-ahead hours—3,037 to 2,217. Consequently, the inventory carrying costs on make-ahead parts is substantially less—$404.15 to $294.84. On the other hand, the purchase items show a substantial increase—50 to 59—and they include items that have a high profit advantage to make over purchase. As a result of buying more purchase hours—7,994 to 11,293—the purchase penalty (profit forgone by purchasing instead of making) more than doubles—$9,233.75 to $23,977.50, even in the program of least penalty. This shows at least for this problem that a 10 per cent hedge involves more than just a proportionate increase in risks and stresses the importance of knowing just what is involved.

A comparison of the specific schedules of Programs 1 and 3 shows that the basic program of make-ahead for the first quarter is the same, allowing for 10 per cent increase in quantity. In Program 3, Parts 9 and 10 are purchased instead of made. The fact that the make and make-ahead are the same for both programs indicates that management can, if they want to, go ahead and take the hedge for the first quarter and then take a second look toward the end of the quarter to see what should be done in the light of what actually happened and the new forecast for the remainder of the year.

In general, the importance of making ahead is shown again by Program 3. In addition, the fact that the program provided is one of least penalty for the circumstances is also important from the standpoint of dollars and cents. The startling difference in the cost of purchase and make-ahead hours—$9,637.90 to $24,272.34—provides a basis for considering the purchase of new machinery The total penalty of $24,272.34 is the difference between being able to make everything in the quarter in which it is required and the *least*-penalty program to take the hedge. This penalty value represents additions to profits that could be obtained if machinery and man power were available to satisfy each quarter's requirements as needed. In effect, this value represents an amount that could be invested in new machinery and returned the first year under the conditions of the program.

3. Revising the fourth-quarter program for unplanned variations

If it develops, as the year progresses, that demand is going to be higher than anticipated, a least-cost program can be worked out to meet the new conditions. For example, assume that there has been an unexpected increase in demand during July and August, and a similar increase is in prospect for the last quarter. In total, let us say that this amounts to a 10 per cent increase over the previous forecast. Calculating a revised program for the last 4 months—September, October, November, and

December—shows that one-third of the hours needed are purchased. For all practical purposes as many parts as possible are made as needed with only 10 hours being made ahead. In general, the same parts are made and purchased as were made and purchased in each of the other programs compared to Program 1.

4. Other alternative programs

Many other variations on the original program can be explored to provide management with information on which to base decisions. Some of the ones that would probably be of value are:

1. Including additional machine capacity to determine the program that would make best use of new machinery and the additional profit that could be obtained.
2. Including the possibility of overtime operation for relief of peak overload to determine whether purchasing is more costly than overtime.
3. Planning a program using a specified portion of available overtime capacity so that cutbacks can be made in the third and fourth quarters without layoffs or purchase-order cancellations.
4. Computing a program based on a constant level of demand. Such a program would be necessary if seasonal fluctuations were to be met by an accumulation of finished-goods inventory. The comparison of this program with Program 1 would be one factor in determining whether such a program was desirable.
5. In the actual case which the Camtor problem represents, an interesting variation was made. There were three different machine groups involved, so there were three problems. The three were interrelated, however, because the three groups of machines were all run by the same crew, and there were not enough men to run all the machines all of the available time.

To determine which of the machine groups should have idle time, the three problems were superimposed and linked together through a balancing dummy which represented the idle time caused by man-power limitations.

GENERAL COMPARISON OF THE VARIOUS LEAST-COST PROGRAMS

The four programs discussed reveal interesting general comparisons for management thinking. Table 12-10 gives a comparison of alternative programs as follows:

1. Irrespective of the program, it is important to start make-ahead items early in the year. Programs can be revised to reflect changes—either upward or downward as the year progresses. But unless the best start is

made, it is difficult to adjust for it later in the year without incurring undue expense.

2. It is necessary to consider only a few parts, once a program has been set up and calculated. This permits concentrated attention to be put on a few important items with resulting benefits.

3. The penalty involved in the various programs represents a profit advantage that could accrue if additional machine capacity were available. The penalty value, then, is one yardstick by which the purchase of new equipment can be justified and evaluated.

4. The hedge on the forecast is taken up for the most part by purchases. By knowing what these are, by part and amount and time needed, it is possible to work out programs with vendors in advance and limit liability in taking the hedge.

5. In all cases, it is not necessarily the items with the least individual carrying cost that are made ahead and carried in inventory. For example, Part 13 (Program 1)—lowest inventory cost in the program—is not made and carried. Similarly, inspection of Program 1 will show other parts treated the same way.

GENERAL USEFULNESS OF LP INFORMATION TO MANAGEMENT •

The primary results of the LP solution provide for the conditions of the problem, a specific program of make, make-ahead, and buy for minimum penalty. Since the cost of deviations from this program can be calculated readily, management has a solid, factual basis for week-to-week scheduling.

The total penalty associated with any of the computed programs provides a measure of the costliness of the limitations that management has imposed. The basic, or "zero," condition occurs when all items are made in the quarter in which they are needed. This would require the machine capacity to be as flexible as the demand, so that it would be capable of making all the items with a positive manufacturing profit advantage in any quarter.

The total penalty of any of the calculated programs represents the price which management pays because the zero condition cannot be met. This provides management with a general indication of the cost of capacity limitations.

In this, as in all LP applications, other values appear that are of equal or even greater importance than the primary solution. The preparation of data for the LP solution required the analysis of certain data that had not been properly analyzed before. This new look at old information gave management an entirely different perspective.

The best example of this was in the purchase-make cost comparison.

For this comparison it was necessary to determine accurately the actual cost of a purchased item—including freight, cost of purchasing, receiving, and inspecting—and the cost of carrying the inventory resulting from the size of the delivery lot, as well as the purchase price. This was compared with the actual cost of the manufactured item, including material (with in-freight and storage), labor, burden, setup, cost of scheduling and shop ordering, and the cost of carrying the inventory resulting from the size of the manufacturing lot. This comparison was far more realistic than the too-often-used purchase price versus standard manufacturing cost.

Further value to management was provided by translating these comparisons into terms that were usable in the problem. Since the problem was the allocation of machine time, the purchase-make cost comparisons were converted into profit advantage per hour of machine operation. This made all of them directly comparable.

A list of items, arranged in descending order of profit advantage, provides management with a sound basis for deciding which items to make and which to purchase.

This type of analysis is not linear programming, and no knowledge of linear programming is required to develop such cost comparisons. It is significant, however, that these figures were not developed until planning personnel, thinking in terms of linear programming, developed them to use in the modi matrix. Using the PA per machine hour as a basis for purchase-make decisions saved the company $54,000 in the first 3 months.

The list of items in descending order of PA per machine hour is also useful for guiding management thinking in long-range planning. The general nature of the list shows the relative economy of the operation in comparison with vendors. For any given forecast, each item on the list represents a certain machine load. The available machine capacity therefore determines how far down on the list items can be made. The profitability of additional machine capacity can be examined in terms of how it shifts the dividing line.

When the PA list is used in the modi matrix to plan for fluctuating demand, linear programming shows that it is most profitable to draw the purchase-make dividing line at about the same level each quarter. Most of the demand fluctuation is then taken up by make-ahead. This occurs because the inventory carrying cost involved is smaller than the increment in penalty incurred by moving from one item on the PA list to another.

Examination of these relationships affords management an excellent perspective in planning and avoids improper emphasis on the various factors.

The results of this LP analysis have also proved to be important to management in evaluating conditions that were not considered in the calculations. For example, management could not place a value on a constant work load. When the cost of achieving it was calculated, however, management was able to decide whether or not they were willing to spend that amount.

These examples illustrate some of the ways in which a linear-programming application such as the Camtor problem can provide sound, factual data for management decision.

CHAPTER 13

Putting Linear Programming to Work: Problems, Possibilities, Predictions

The growing number of articles, seminars, and panel discussions devoted to linear programming indicates a widening interest in its use in business.

It is a large order, however, to go from interest to actual use. On the way there are many problems and obstacles that must be met successfully if management people are to realize the fullest benefits that are possible to them and their firms. Fortunately, experience in application has made it possible to identify most of the problems and obstacles and meet their challenge successfully.

PROBLEMS IN INTRODUCING AND APPLYING LP

As with anything new, quite a number of people will tend to resist the use of LP primarily because it is strange. This situation is likely to be worse than it would be otherwise because of the term Linear Programming. As we said in the Introduction, the name gives no clue to management people about what it is, where it can be used, or its potential for improving management practice, information, and decisions. Unfortunately, we are stuck with the name, a name that conveys no meaning or message to most people in business. Consequently, there is a very real communications hurdle to overcome at the present level of understanding and acceptance.

The communications hurdle, and other problems involved in putting LP to work in business, should be less difficult for firms that have a functioning OR group than for those that do not have such a group. It is quite likely that LP has been discussed and used in a number of such firms, so that management people have had some exposure to discussion, even if they have not been in on the application. Of course, where application has been successful, the problems of getting understanding, acceptance, and additional areas for application are much easier to solve.

The same situation is likely to be true—perhaps to a lesser extent—in

firms in which people are applying LP to specific areas without a formal OR setup. But where little or nothing has been done—with the possible exception of attending a seminar or reading an article—introducing and applying LP will require careful planning to be effective.

It is to the people in the last category that most of the following suggestions for putting LP to work are directed. Those in the first two categories also may find a useful suggestion or two.

Experience so far indicates that to get the most out of LP requires the solution of two major groups of problems. First are the direct problems of providing the personnel, the know-how, and the data for setting up and solving problems. Second, to attain the most benefits from the use of LP, it must be assimilated into the organization as a working tool and as an integral part of management thinking.

Problems in the first group pose some difficulty, but they can be solved by staff specialists trained for the purpose. The second group is more troublesome and requires attention to matters of organization, communications, and long-range planning. In fact, solving the second group of problems may well depend on having used the proper approach to the first group.

It is possible, for example, to start into LP by having a consultant or other outside authority come in, study a problem, develop data, and work out a solution. This approach would provide a solution to a specific, current problem. It would not, however, help the company, except perhaps in a limited way, to use LP in the future. Neither would it provide a basis for integrating LP into the thinking of the management group.

Even in the first attempts at applying LP, it is essential that the analysis of the problem, the application of data, and the interpretation of results is done by those in the organization who are normally responsible for these activities. In this way, LP becomes a tool of the management organization rather than a gimmick for a one-shot demonstration. If consultants are used, their function should be teaching, coaching, and coordinating—not doing the work—if the firm is to obtain the most effective results in the long run. There may be some cases where this is not true, but they will be the exception rather than the rule.

In getting started, then, management can lay a good foundation for future benefits by making sure that LP know-how is brought into the working organization through good training and practice.

It is best to have several people trained in LP even though only a few are to spend full time working on LP problems. The more widespread the knowledge of LP is in the beginning, the easier will be the gathering of information and selling of results. Furthermore, having several trained people will enable the person who is tackling a problem to get advice, assistance, and the benefit of group thinking when necessary.

When people are being selected for training, it is possible to minimize future data-gathering difficulties by including personnel from the areas of the organization which will have to furnish data. For example, problems of getting cost information in usable form are greatly simplified if one or more cost accountants and industrial engineers have had LP training.

1. Using LP as a management tool

Proper training of selected personnel and the impetus provided by the interest of an executive with authority in the area of the problem are the basic ingredients of a first linear-programming application. A successful first project will do much toward developing interest and initiative in further application. When this happens, it is important that management be prepared to capitalize on this momentum to expand the area of LP applications. To do this, management must cope with the problems in the second group. These problems fall generally into three areas:

1. Fitting LP into the organization
2. Developing awareness and use of LP
3. Communicating effectively with others

By careful planning to meet these problems, management will assure maximum benefits from their efforts to introduce and apply LP.

2. Fitting LP into the organization

Since LP is a method for evaluating pros and cons that assist in decision making, the authority and responsibility for decisions must be considered when fitting the LP activity into the organization. There is no one organization arrangement that is "right" for LP. The fact that the LP specialist needs to have a relatively free hand in examining problems and gathering data and must often work across department and division lines has led some companies to put LP work directly under one of the general executives.

One of the advantages of this location is that it makes it easier to obtain needed information and gives status and importance to projects that are undertaken. Fitting LP into the organization in this way seems to be confined to the larger firms—those generally having OR groups. Another advantage of the central location is that larger problems with larger potential are more likely to be undertaken under this arrangement than when the activity is located further down the management ladder. Frequently, the broader problems and applications require greater time to work out because of their size, and the results may be a longer time in coming. This will not be a problem if management is willing to wait for results or considers the application effort as an investment in business

research which may or may not pay a return. Quite frequently a management geared to day-to-day activity may become impatient for results if they are not forthcoming in a relatively short time. If this will be a problems, it may be well to consider an application that can be confined to one problem area as a door opener. To do this, many firms have set up LP activities as part of a manufacturing, production-control, sales-planning, or other functional area. Under this arrangement the LP work tends to be confined to problems within the functional area but serves the purpose of demonstrating the potential without "waiting a year" for results to appear.

Much of the application to date has been in these functional areas—and with considerable benefit. The selection of more specialized areas, such as production planning or purchase-make, has resulted in highly satisfactory results in a relatively short time. This approach has the advantage of providing a profitable application with measurable results without large investments in time and talent. A successful application, even a small one, makes it easier to extend and expand LP and OR to other areas and problems than if there were no results to point to. This is particularly true when management can use the savings or profits to underwrite or finance the other projects. The old adage "Nothing succeeds like success" applies to LP applications as well as to other situations.

Irrespective of whether a large-scale or small-scale application is made, the people who do the LP work should serve as staff specialists to the line organization. LP people provide information. The line people have to do something about it if results are to be obtained.

3. Introducing LP to the organization

Acceptance and use of anything new is directly related to how well it is understood by the people who are likely to come into contact with it. For this reason, priority and attention should be given to acquainting people within the organization with LP and its possible uses for helping them and the firm.

One way to do this is to have competent people make a survey to determine where and if LP can be used. The survey should point out the areas of application, the information that is needed, the priority and order in which installation steps are to be taken, an estimate of the possible results and benefits that can be anticipated, and the time and costs involved. On the basis of the findings of the survey, management can then decide whether or not to go ahead with using and installing LP.

There have been a number of situations in which management was sufficiently convinced of the potentialities that a decision was made to go

ahead without making a survey. Under these circumstances, the first step in introducing LP is to acquaint the organization with it and equip selected people to use it. Generally, this procedure is carried out in two steps. One step involves "technical" training for those people who will *do* the work. The other step involves "appreciation" training for two groups of people, those who will *supply* information and those who will *use* the information to assist them with planning and decision making. Providing appreciation-type meetings or sessions for those expected to supply information will enable them to cooperate more effectively than they otherwise could. Holding appreciation sessions for executives and managers will enable them to see the logic of LP, understand what LP people do and the information they need to do it, and recognize possible applications of benefit to the firm which may never be apparent to people lower in the management echelon. All these activities will increase the confidence of the executive and manager in the information provided by the LP people, which means that it is more likely to be considered and used when decisions are made. Certainly an understanding and awareness of the uses, limitations, and potential of LP can be a powerful force for its adoption and application. It will also tend to eliminate defensiveness, if any, on the part of those people who might look upon the improvement from using LP as a criticism of their way of operating and managing. When this kind of awkwardness comes up, either visibly or invisibly, and it will from time to time, one of the following suggestions may help to smooth the ruffled feathers.

1. Stress the *newness* and the progress and benefit that have come from trying new methods and developments in the past.
2. Point out that LP is a powerful new tool that will provide information that simplifies the work of organizing information and making decisions.
3. Explain applications and examples of the use of LP in other firms having similar problems and the *benefits* that resulted.

4. Communicating effectively with others

The safest assumption to make in applying LP at this stage of development and acceptance is to assume that there will be a communications problem. This is likely to be as true in firms that have had OR experience as it is in firms that have not.

The basic problems of communications come from the people who know and do linear-programming work and who use words in talking about LP that people in business may not understand. Management people, on the other hand, are not accustomed to talking and thinking in terms of LP mathematics, and, unfortunately, the LP man often is not

well enough versed in business terminology and thinking to make his points clear to management in management terms. Bridging the communications gap and developing a common language and understanding is an essential step for enduring and successful applications. Of course, the training will help to overcome the problem, as will successful application. The essential first step, however, is to recognize that a communications problem does exist and then take the steps to meet it head on. Meetings, training sessions, discussions, and question periods will all help to introduce LP to the organization.

POSSIBLE APPLICATIONS: WHERE TO START

People in business will find many uses for linear programming as they become better acquainted with its potentialities and possibilities. Each working installation will encourage others. Because the use of LP strengthens a firm's competitive position, other firms learning of their competitor's progress will be forced to keep abreast of developments and applications.

The applications discussed in the preceding chapters, particularly in Chapters 10, 11, and 12, point to some of the problems that management people have tackled successfully with considerable benefit to their firms. In Chapter 1 we pointed out many areas in which successful application has been made. For convenience, they are repeated here because they represent places at which a start can be made.

1. Determining most profitable product mix to be obtained from existing facilities
2. Determining which parts to make and which to buy to obtain maximum profit margin
3. Scheduling orders to machine centers at least cost consistent with delivery promises and schedules
4. Establishing best location of warehouses to minimize transportation costs
5. Selecting equipment and evaluating methods improvements that maximize profit margin
6. Planning profits on a fiscal-year basis to maximize net return on an investment in plant, facilities, equipment, cash on hand, and inventory
7. Supplying a fluctuating sales demand at least inventory cost considering a fixed level of production and stabilized employment
8. Allocating production releases among several plants so as to maximize profit margin, considering manufacturing and distribution costs
9. Determining equitable sales and incentive compensation
10. Determining the feed mix that satisfies nutritional requirements and minimizes the cost of raising livestock

11. Programming a chemical-distilling-type operation, including the processing of purchased material, to obtain the highest manufacturing margin with limits of sales demand

12. Planning the most profitable combinations of sales requirements and plant capacity that obtain a fair share of the market

These cases in no way describe the full potential of the technique. There are many other fields in which LP has been and will be used effectively. The available literature indicates that, in addition to manufacturing, applications have been made in the areas of transportation, mining, personnel, and agriculture. In the last case, excellent ideas have developed for allocating land to crops and developing feed mixes. We can expect to see additional applications in all these areas in the future.

Because LP is particularly effective where problems are large and data are difficult to organize into a pattern, governments may find use for it in deciding how to allocate a part or all of the nation's productive resources. It seems possible that LP can be used not only to decide how to allocate present resources effectively, but also to indicate what elements should be expanded to provide greatest gain. For example, it may be possible to develop a program of exports and imports. Those exports could be selected that were worth more to some other nation than they were to the exporting nation. Likewise, the technique could be used to select the imports that can contribute the most to the economy of the nation.

1. Selecting the first LP application

The problem selected as a place to start LP application will have considerable influence on the over-all success. Care should be exercised in selecting the problem because it will demonstrate the practicality and usefulness of the technique, provide a training ground for learning, and demonstrate methods of gathering data.

Some of the characteristics of the right starting problem are as follows:

1. It must be generally recognized as a problem that needs to be solved. This recognition should be gained before the work of gathering data and solving the problem begins. It will be difficult to gain acceptance of the solution of a problem that only a few people recognize as a problem.

2. The probable benefits arising from the solution of the problem should be easily recognized and understood.

3. The methods used in gathering and organizing information should establish a pattern that can be followed in similar work on other problems. In other words, the work on the first problem should be an example of the methods that can be used in the solution of other problems.

4. The problem should be sufficiently complex to make it unreasonable

to expect that the best solution of the problem can be found by intuitive reasoning.

These characteristics of the first problem are in addition to rules for recognizing LP problems given in Chapter 8, "Recognizing Problems Which Can Be Solved by Linear-programming Methods." Giving special attention to the first problem will do much to get the linear-programming work under way with a maximum of acceptance and understanding. The possible difference in the long-range results, if any, far more than justifies the extra time and care required for a careful selection of the first problem and application.

PREDICTIONS: THE FUTURE OF LP

Simply stated from the standpoint of the firm, LP is one of the management tools that aid in improving the performance of the functions of a business. The competitive situation makes it necessary for a nation or a firm to do everything possible to maintain and improve competitive position. On this basis, it would appear that linear programming will move forward as a part of the increased use of scientific and mathematical tools rather than as a particular technique. It will, in due course, be recognized and accepted as a tool of scientific management.

1. The increased use of OR will increase the use of LP

The past few decades have seen an increasing use of tools developed by science in government and business activities. This trend has contributed much to our higher standard of living. Since World War II. some companies have applied the combined power of several sciences to problems through the use of operations-research groups. Such a group might consist of a metallurgist, a psychologist, an operating man, and a financial man. These men seek the aid of physicists, chemists, engineers, mathematicians, and other specialists whenever it seems that they might aid in solving the problem at hand. The accomplishments of OR groups indicate that this method of combining the power of several disciplines to solve one problem is effective and will see wider use in the future. As the number of such groups increases, we can expect to see the use of LP increase.

2. The use of LP for solving operating problems will increase

The possibilities are good that there will also be an increase in the use of linear programming in solving specific operating problems. Management people will learn of uses for solving such problems as scheduling orders to machines, planning the transportation of material from a number of origins to a number of destinations, or solving the make-or-

buy problem. As the management of competitive firms learn of the economies that have been effected, they will begin to use linear programming in their firms also.

3. Educational institutions will expand the fields of LP

Quite likely there will be an increase in the number of linear-programming courses offered in schools and colleges. Educators will learn more concerning the principles and applications of LP methods and techniques. Investigations in the programming field should bring further advances. For example, the broad field in nonlinear programming that has only been partially explored is a subject for academic research.

As more and better training in the use of linear programming becomes available more trained people will be available. These people will see problems that linear programming can solve, and they will point out the advantages of using LP and recommend its use for solving these problems.

4. Electronic computers will expand the use of LP

The increasing use and availability of electronic computers should also tend to expand the use of linear programming. These computers provide a way of solving large simplex matrices that it would be impractical to solve in any other way. Although the development of simplified methods, such as the modi, ratio-analysis, and index-number methods, provides for solving a wide range of problems with desk calculators, there are applications of linear programming where an electronic computer is the only practical way of solving the problem.

The number of electronic computers in use is increasing rapidly. It is logical to expect this increase to bring about an increase in the number of problems that can and will be solved by the use of linear programming.

CONCLUSION

The record for LP is reasonably clear.

On the basis of results, it is proving to be a useful and valuable aid to management people. Its value, as we have said, lies in two areas: assisting executives and managers to be more effective in using resources under their direction and providing a quicker insight into business problems for the younger management man, thereby assisting him to develop.

We know there are areas and problems in which LP now has little or no value. We cannot define these areas explicitly because every day new developments and applications are pushing back the frontiers and increasing its use. The development of nonlinear programming and game theory are cases in point. We do know that certain requirements must

be met for proper application of LP methods at the present state of development. That is why particular attention and emphasis was given in Section II, "Methods," to the requirements for using each of the methods.

The fact that some problems do not yield to LP does not lessen its importance in the areas where it can be applied. As management people, we are obligated to use our time, talents, money, machines, material, man power, and other resources as effectively as we possibly can. The objective of this book is to provide information to assist in accomplishing that goal.

Bibliography

This bibliography is intended to present a cross section of the reading that is available from various sources on the subject of linear programming. For convenience, the references are grouped into two main categories—"Books and Proceedings" and "Papers, Articles, and Reports"—and are listed within the group in order of publication date.

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SECTION FOUR

Technical Appendixes

The purpose of these appendixes is to present some of the mathematical concepts and developments which underlie the computations used in the preceding sections.

There are two appendixes Appendix A is entitled "Mathematical Explanation of the Simplex Method for Solving Linear-programming Problems" Appendix B is entitled "Comparison of the Simplex Method and the Modi Method." We believe that each of these sections will add to the completeness with which we have presented linear programming, although they are not essential to a use of the technique. Appendix A, particularly, will have special appeal for those readers and students interested in knowing more about why the computations produce the best answer. By presenting additional information about the relationship of the two most important computational methods, Appendix B will add to their understanding and increased use.

As a suggestion, we recommend that the management-oriented man read the preceding sections before investigating this section. The technical man and the student will perhaps find it to their advantage to read this section before proceeding with the first three sections.

We make no claim of authorship for the material presented in this section, which has been collected and summarized from the various references mentioned throughout the book. The material in Appendix A has been reproduced with the permission of Mr Kurt Eisemann, Research Mathematician, IBM Corporation. For additional information the reader is referred to:

1. "Linear Programming," *Quarterly of Applied Mathematics*, vol. 13, no. 3, October, 1955, page 209.
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It is to the authors of the cited references that the credit must go for the other material and information contained in this section.

APPENDIX A

Mathematical Description of the Simplex Method for Solving Linear-programming Problems

The general linear-programming problem is one of determining, out of an infinite number of possible solutions, that unique solution which makes a certain function (profit margin, cost, utilization, or time) a maximum or a minimum. The computed programs provide the best possible planning information on which operations may be based under the specified restrictions, limitations, and conditions as set forth in the problem.

Restrictions or constraints which limit results generally take the form of inequalities with nonnegative variables. Whenever the inequalities encountered are linear in the variables, the problem is said to be a "linear-programming problem."

The correct formulation of problems expressible in linear-programming form is fundamental to obtaining a useful and valid answer. The solution can be calculated fairly easily in a routine fashion once the problem has been formulated correctly. Elaborate care is essential to ensure that none of the restrictions, limitations, and conditions are overlooked.

The derivation of the simplex method, which is the prime computational method of linear programming, will be developed in this section. To make the derivation easier to follow, a small example problem will be solved step by step as we progress through the derivation.

Because we want the derivation to proceed in an orderly way, it will be discussed in turn under the following headings:

1. Expressing the problem in the required mathematical form (expressing the desired objective in equation form and expressing the restrictions, conditions, and limitations as equations and inequalities)
2. Converting inequalities to equivalent equalities
3. Setting up a unit basis
4. Changing the basis
5. Transforming to a new basis
6. Restoring the basis to unit form
7. Calculating subsequent iterations

Each of these steps will be discussed in turn in the pages which follow. In addition, a brief discussion of degeneracy, convergence and duality is included at the end of this section.

1. Expressing the problem in the required mathematical form

The mathematical formulation of a linear-programming problem is as follows:

Find values for $x_1, x_2, \ldots x_{\bar{n}}$ which satisfy the conditions

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1\bar{n}}x_{\bar{n}} &\leqq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2\bar{n}}x_{\bar{n}} &\leqq b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{m\bar{n}}x_{\bar{n}} &\leqq b_m \end{aligned} \tag{A-1}$$

$$x_j \geqq 0 \text{ for all values of } j \tag{A-2}$$

such that

$$f = c_1x_1 + c_2x_2 + \cdots c_{\bar{n}}x_{\bar{n}} = \text{a maximum} \tag{A-3}$$

where a_{ij}, b_i, and c_i are given constants

When we express the inequalities of Equation (A-1) in a more general form we obtain the expression

$$\sum_{j=1}^{\bar{n}} a_{ij}x_j \leqq b_i, \; i = 1 \ldots m \tag{A-4}$$

When we express the functional Equation (A-3) in a more general form we obtain the expression

$$f = \Sigma c_jx_j = \text{maximum} \tag{A-5}$$

When we express the inequalities of Equation (A-1) as a coefficients matrix, denoting scalars by lower case letters and denoting the columns by $A_1, A_2, \ldots A\bar{n}, B$, we obtain the expression

$$A_1x_1 + A_2x_2 + \cdots A_{\bar{n}}x_{\bar{n}} \leqq B \tag{A-6}$$

The expressions shown under Equation (A-1) are inequalities in which the right side of the expression is either greater than or equal to the left side. Equation (A-1) may include *inequalities* of the form

$$a_{31}x_1 + a_{32}x_2 + \cdots \geqq b_3 \tag{A-7}$$

$$\sum_{j=1}^{\bar{n}} a_{ij}x_j \geqq b_i \tag{A-8}$$

or may include *equations* of the form

$$a_{41}x_1 + a_{42}x_2 + \cdots = b_4 \tag{A-9}$$

$$\Sigma a_{ij}x_j = b_i \tag{A-10}$$

where the general notation has the following meanings:

x = variable whose value is to be determined by the calculations
$x_{\bar{n}}$ = general notation representing the last of a group of variables
x_j = general notation representing an individual variable
a = constant which is given and which multiplies a variable
a_{ij} = general notation representing a constant which multiplies a variable in the ith row and jth column
b = constant which is given
b_i = general notation representing an individual constant
m = number of rows in a coefficients matrix which is an orderly arrangement of numbers
n = number of columns in a coefficients matrix
f = general notation representing the objective or functional equation
c = coefficients which are given and which multiply the various x's in the functional equation
$c_{\bar{n}}$ = general notation representing the constant by which the last $x_j(x_{\bar{n}})$ is multiplied
A = matrix or orderly arrangement of coefficients
A_j = jth column of a coefficients matrix
B = general notation for right-hand-side constants in a coefficients matrix

If we are to solve for a minimum instead of a maximum, the requirements $\phi = \Sigma d_j x_j$ = a minimum can be converted by substituting $d_j = -c_j$ in the functional Equation (A-5) $f = \Sigma c_j x_j$. When this is done we obtain the expression

$$f = -\phi = \Sigma(-d_j)x_j = \Sigma c_j x_j = \text{maximum} \qquad \text{(A-11)}$$

At this point it will be helpful to express the general notation in specific terms by referring to an example problem. The problem, which will be brought into the derivation as it proceeds, is a problem of determining the product mix of highest profit margin under the conditions, restrictions, and limitations given in Table A-1.

Applying this general formulation and notation to the conditions and restrictions of the problem, we obtain the following arrangement:

$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &\leqq 48 \\ 4x_1 + 2x_2 + 3x_3 &\leqq 60 \\ 3x_1 + x_3 &\leqq 36 \\ x_2 &= 5 \\ x_1 &\geqq 0 \\ x_3 &\geqq 0 \end{aligned}$$

$$f = 6x_1 + 4x_2 + 3x_3 = \text{maximum profit margin}$$

Table A-1. Production and Profit Information

Machine centers	Hours per piece			Machine time available (in hours)
	A	B	C	
I	2	4	3	Up to 48
II	4	2	3	Up to 60
III	3	0	1	Up to 36
Profit margin per piece (in dollars)	6	4	3	

Assumptions:
It is possible to sell everything that is made.
It is necessary to produce exactly five pieces of B to satisfy a good customer.

2. Converting inequalities to equivalent equalities

The inequalities of Equation (A-1) must be converted into equalities with nonnegative right-hand numbers by use of the following rules:

1. Multiply by -1 when necessary to make all $b_i \geqq 0$ (A-12)

2. Multiply by -1 when $b_i = 0$ and when it is desirable to convert inequalities $\Sigma a_{ij}x_j \geqq 0$ into the form $\Sigma a'_{ij}x_j \leqq 0$ (A-13) this will provide a place for natural unit vectors to enter.

3. Replace inequalities of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots a_{i\bar{n}}x_{\bar{n}} \leqq b_i$ with equivalent equalities of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{i\bar{n}}x_{\bar{n}} + x_{\bar{n}+i} = b_i \tag{A-14}$$

4. Replace inequalities of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots a_{i\bar{n}}x_{\bar{n}} \leqq b_i$ with equivalent equalities of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{i\bar{n}}x_{\bar{n}} - x_{\bar{n}+i} = b_i \tag{A-15}$$

The variables $x_{\bar{n}+1}$ are termed *slack variables* because they take up the slack by which the inequality is permitted to differ from the equivalent equality.

Slack variables have the following characteristics:

1. Must be nonnegative:

$$x_{\bar{n}+i} \geqq 0 \tag{A-16}$$

2. May appear in the final solution but do not make a positive contribution to it—i.e., do not contribute to the value of the functional

3. Are not used when the original expression is an equation

With the inclusion of slack variables, the general formulation of the problem is as follows:

$$A_1x_1 + A_2x_2 + \cdots A_{\bar{n}'}x_{\bar{n}'} = B \tag{A-17}$$

where $b_i \geqq 0$

$$x_j \geqq 0 \text{ for all } j \tag{A-18}$$

$$f = c_1x_1 + c_2x_2 + \cdots c_{\bar{n}'}x_{\bar{n}'} = \text{maximum} \tag{A-19}$$

Underlying definitions and theorems

Before proceeding further to describe the mathematical derivation of the simplex we must establish certain definitions, theorems, and considerations that underlie the method. The following definitions apply:

1. A *feasible solution* contains a set of x_j which satisfy Equations (A-17) and (A-18) but not necessarily (A-19) (a solution does not have to be maximal to be feasible).
2. A *basic feasible solution* is a feasible solution with not more than m nonzero values of x_j.
3. A *maximal solution* must satisfy Equations (A-17), (A-18), and (A-19).
4. A *basic maximal solution* is a maximal solution with not more than m nonzero values of x_j.

Based on these definitions, the following mathematical theorems apply:

Theorem 1: Whenever a feasible solution exists, a basic feasible solution also exists.

Theorem 2: If the functional expression remains finite for all possible feasible solutions, there exists a basic maximal solution.

By means of the foregoing definitions and theorems, the solution to a linear-programming problem by the simplex procedure is obtained as follows:

1. Construct a basic feasible solution.
2. Replace the first basic feasible solution by another basic feasible solution which is an improvement in f (or at least equal to) to the previous solution (f).
3. Continue the replacement process until certain criteria indicate that no further increase in f is possible, or certain characteristics in the computational procedure indicate that $f_{\max} = \infty$. Each solution replacement by a new solution is called an *iteration*. In the simplex procedure there is always a finite number of iterations.

3. Setting up a unit basis

If the first m column vectors A_j are linearly independent, they provide a basis in terms of which all columns of the coefficient matrix can be expressed as:

$$A_j = d_{1j}A_1 + d_{2j}A_2 + \cdots d_{mj}A_m \tag{A-20}$$

Further, if we can arrange it that $A_1, A_2, \ldots A_m$ are the m columns I_i of the mth-order identity matrix (unit columns forming a unit basis), we obtain a basic feasible solution as follows:

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix} = a_{1j} \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + a_{2j} \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + a_{mj} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{A-21}$$

where $d_{ij} = a_{ij}$ as given

$$x_i = b_i \geqq 0 \ (i = 1 \text{ to } m) \text{ We arbitrarily set } x_j = 0 \text{ for } j > m \tag{A-22}$$

In setting up a unit basis we are in effect setting up a square diagonal of ones. Some of the ones may come from the original A_j columns which may be unit columns. Each slack variable $x_{\bar{n}+i}$ supplied by $\leqq$ introduces a unit column $A_{\bar{n}+i}$ containing a one. Any inequality of the type $\geqq$ introduces a slack vector $-I_j$ into the matrix of coefficients and therefore will require the introduction of an artificial vector $+I_i$. The artificial variable which has the general form $X_{\bar{n}'+i}$ also contributes to the unit basis so that the coefficients matrix A contains all possible m unit columns I_j.

We stated that Equations (A-1) permit a certain amount of slack which may appear in the final solution. Because they make no contribution to the profit margin function f, the c_j profit coefficients corresponding to *slack* are assigned a value of zero.

The *artificial* variables, however, are introduced only for convenience in getting started (we need a square unit basis) and have no place in the final solution. To make certain that these variables do not appear in the solution, we attach an unduly large penalty to them. By assigning a large penalty $-M$ (a very large negative number) to the coefficients c_j, we shall ensure that f cannot possibly reach its maximum as long as any feasible solution contains any of the artificial variables. These variables are eliminated by meeting the requirements of the computational procedure, which states that a maximum has not been reached as long as there are negative values in the base row of an iteration.

Once the columns of the unit basis have been established, we rearrange the A columns so as to place the identity submatrix on the left. It is necessary to keep the values in each column intact in this rearrangement. The purpose of the rearrangement is to simplify the calculations and display the program at each iteration.

Under the rearrangement the general formulation of a linear-programming problem for solution by the simplex method is as follows:

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = B \tag{A-23}$$

$$A_j = I_j \text{ for } j \leqq m$$

$$b_i \geqq 0 \qquad i = 1 \ldots m \tag{A-24}$$

$$x_j \geqq 0 \qquad j = 1 \ldots n$$

$$f = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \text{maximum} \tag{A-25}$$

We can obtain an initial basic *feasible* solution from Equation (A-23) in which $x_i = 0$ for i greater than m and in which $x_i = b_i$ for $i \leqq m$ which satisfies

1. $$A_1x_1 + A_2x_2 + \cdots + A_mx_m = B \tag{A-26}$$

where $A_j = I_j$ for all j

2. $$f = c_1x_1 + c_2x_2 + \cdots + c_mx_m \tag{A-27}$$

which is not necessarily a maximum.

When we convert the inequalities of the example problem to equivalent equalities, we obtain the following relationships:

$$2x_1 + 4x_2 + 3x_3 + x_4 = 48$$

where x_4 = slack variable permitting an equivalent equality

$$4x_1 + 2x_2 + 3x_3 + x_5 = 60$$

where x_5 = slack variable permitting an equivalent equality

$$3x_1 + x_3 + x_6 = 36$$

where x_6 = slack variable permitting an equivalent equality

$$x_3 + x_7 = 5$$

where x_7 = artificial variable required by the computations

When we establish a unit basis and rearrange columns, we obtain the following arrangement and initial basic feasible solution:

$$\begin{array}{c} \begin{array}{ccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 4 & 3 \\ 0 & 1 & 0 & 0 & 4 & 2 & 3 \\ 0 & 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \\ x'_5 \\ x'_6 \\ x'_7 \end{bmatrix} = \begin{bmatrix} 48 \\ 60 \\ 36 \\ 5 \end{bmatrix}$$

in which we establish the following relationship:

$$x_1' = x_4 \qquad x_4' = x_7 \qquad x_6' = x_2$$
$$x_2' = x_5 \qquad x_5' = x_1 \qquad x_7' = x_3$$
$$x_3' = x_6$$

The resulting functional or profit-margin equation is as follows:

$$f = 0x_1' + 0x_2' + 0x_3' - Mx_4' + 6x_5' + 4x_6' + 3x_7' = \text{maximum}$$

The initial basic feasible solution is as follows:

$$x_1' = 48 \qquad x_3' = 36 \qquad x_5' = 0 \qquad x_7' = 0$$
$$x_2' = 60 \qquad x_4' = 5 \qquad x_6' = 0$$

4. Changing the basis

The initial basic feasible solution involves nonzero variables x_i for $i = 1$ to m. The next step is to improve f. This is done by establishing x_r $(r \leqq m)$ and some value $x_k \geqq 0$ $(k > m)$ such that replacement of the term $A_r x_r$ in Equation (A-23) by the term $A_k x_k$ (A_k is given) will leave all conditions satisfied and at the same time will result in an improvement in f. Under this procedure the new solution will have the following characteristics:

1. Involve only m variables x_i
2. Be a basic feasible solution
3. Represent an improvement in f

We can express the same process and relationship in terms of the given vectors A_j instead of the variables x_i as follows:

1. Replace A_r $(r \leqq m)$ one of the unit base vectors by A_k $(k > m)$ with one of the matrix columns not in the basis.

2. Determine the corresponding r and k such that Equations (A-23) and (A-24) will be satisfied and f will increase.

This replacement process is called "introducing vector A_k into the basis" and proceeds as follows:

Referring to Equation (A-21) for $j = k$ we obtain

$$A_k - (A_1 a_{1k} + \cdots + A_m a_{mk}) = 0 \tag{A-28}$$

Introducing A_k into the basis we add θ times identity Equation (A-28) to Equation (A-26), obtaining

$$A_k\theta + A_1(x_1 - \theta a_{1k}) + \cdots + A_r(x_r - \theta a_{rk}) + \cdots + A_m(x_m - \theta a_{mk}) = B \tag{A-29}$$

For $i = k$, restriction $x_i \geqq 0$ applied to Equation (A-29) requires that

$$\theta \geqq 0 \tag{A-30}$$

Constructing a new f corresponding to Equation (A-29), we obtain

$$f_j = c_1a_{1j} + \cdots + c_ma_{mj} \qquad (j = 1 \text{ to } n) \tag{A-31}$$

Substituting $j = k$, subtracting Equation (A-31) from Equation (A-27), and adding the term $c_k\theta$ to each side of the expression we obtain

$$f' = f - \theta(f_k - c_k) = c_k\theta + c_1(x_1 - \theta a_{1k}) + \cdots + c_m(x_m - \theta a_{mk}) \tag{A-32}$$

The right-hand side is the new f for Equation (A-29), and for $\theta > 0$ we can make $f' > f$ only if k is so chosen that

$$f_k - c_k < 0 \tag{A-33}$$

When, after a finite number of iterations, a stage is reached such that no negative $f_k - c_k$ remains, no further increase in f is possible, and the best or maximal solution has been reached in the calculations.

Choosing k

Theoretically, the largest increase in f for one iteration is obtained when $-\theta(f_k - c_k)$ is as large as possible, but it is more convenient to use the following selection procedure.

If we substitute the expression

$$g_j \equiv f_j - c_j = c_1a_{1j} + \cdots + c_ma_{mj} - c_j \; (j = 1 \text{ to } n) \tag{A-34}$$

for Equation (A-31), we can examine only the negative

$$g_j \equiv f_j - c_j \tag{A-35}$$

select the most negative one, and use that j as k.

Choosing r

For all $i \leqq m$ requirement $x_i \geqq 0$ applied to the coefficients of A in Equation (A-29) becomes the following:

$$x_i - \theta a_{ik} \geqq 0 \tag{A-36}$$

For an $a_{ik} \leqq 0$, this requirement is automatically satisfied for any $\theta \geqq 0$ because

$$x_i \geqq 0 \tag{A-37}$$

For an $a_{ik} > 0$, we must make $\theta \leqq \dfrac{x_i}{a_{ik}}$ for all

$$i \leqq m \tag{A-38}$$

Elimination of A_r requires $x_r - \theta a_{rk} = 0$. For some r, $\theta = \dfrac{x_r}{a_{rk}}$

For the chosen k, select only those a_{ik} which are > 0 to form the quotients

$$q_i = \frac{x_i}{a_{ik}} \qquad (\geqq 0)$$

Select r = the i corresponding to the smallest q_i. Therefore

$$\theta = \frac{x_r}{a_{rk}} = \min_{a_{ik} > 0} \frac{x_i}{a_{ik}}$$

When we change the basis, choose an r and k using our example problem, and rearrange the matrix for ease of computation, we obtain the following:

c_i of basis	Basis \ c_j vector	0	0	0	0	$-M$	6	4	3
		$B = A_0$	A_1	A_2	A_3	A_4	A_5	A_6	A_7
0	A_1	48	1	0	0	0	2	4	3
0	A_2	60	0	1	0	0	4	2	3
0	A_3	36	0	0	1	0	3	0	1
$-M$	A_4	5	0	0	0	1	0	1	0
$g_j = (\sum_i c_i a_{ij} - c_j)$		$-5M$	0	0	0	0	-6	$-4-M$	-3

The table gives the following information:

$$f = -5M \qquad k = 6(A_6)$$
$$r = 4 \qquad \theta = 5/1 = 5$$

5. Transforming to a new basis

Changes to the basis consisted in introducing A_k into the basis and removing A_r. Each A_j expressed in terms of the old basis must now be converted into a new form A_j', which similarly refers to the new basis.

From Equation (A-21) for any j we obtain

$$A_j = a_{1j}A_1 + \cdots + a_{rj}A_r + \cdots + a_{mj}A_m \tag{A-39}$$

For $j = k$, we obtain

$$A_k = a_{1k}A_1 + \cdots a_{rk}A_r + \cdots + a_{mk}A_m \tag{A-40}$$

Solving for A_r we obtain

$$A_r = \frac{1}{a_{rk}}\left(A_k - \sum_{i \neq r} a_{ik}A_i\right) \tag{A-41}$$

where the term $i = r$ must be excluded under the summation.

This expresses A_r in terms of the new basis and when substituted into Equation (A-39) for the one term containing A_r will express any A_j in terms of the new basis as follows:

$$A_j' = \frac{a_{rj}}{a_{rk}} A_k + \sum_{i \neq r} \left(a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik} \right) A_i \tag{A-42}$$

This transformation also applies to the vector $B \equiv A_0$.

For $j = k$ in Equation (A-42) we obtain

$$A_k' = 1 \cdot A + \Sigma 0 \cdot A_i \tag{A-43}$$

6. Restoring the basis to unit form

The new basis is no longer a unit basis but instead contains

$$A_k = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix}$$

A notational device, however, will restore the basis to unit form as follows:

1. In Equation (A-21) the rth component of any vector indicated the coefficient of A_r, which henceforth is zero.

2. We must adjoin to Equation (A-21) an additional term $a_{kj}'A_k'$ and utilize the vacancy of the rth component of a_{rj}' of each vector to indicate $a_{kj} = \dfrac{a_{rj}}{a_{rk}}$.

3. This new notation becomes

$$A_k' \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = I_r \tag{A-44}$$

rth component

which is now in unit form.

For the other unit vectors A_j remaining in the basis $(j \leqq m, j \neq r)$ we have $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = 1$ when $i = j$.

Their new form is

$$A_j' = 0 + \Sigma_i(a_{ij} - 0)A_i = A_j = I_j \qquad (j \leqq m, j \neq r) \qquad \text{(A-45)}$$

They are thus completely unchanged by the transformation. The former unit-base vector A_r $(j = r)$, however, becomes

$$A_r' = \frac{1}{a_{rk}} A_k + \sum_{i \neq r} \left(-\frac{a_{ik}}{a_{rk}}\right) A_i \qquad \text{(A-46)}$$

which is no longer in unit form.

With our notation Equation (A-42) becomes the following:

$$A_j' = \begin{bmatrix} a_{1j}' \\ \cdot \\ \cdot \\ \cdot \\ a_{mj}' \end{bmatrix} = \frac{a_{rj}}{a_{rk}} I_r + \sum_{i \neq r} \left(a_{ij} - \frac{a_{rj}}{a_{rk}}\right) I_i \qquad \text{(A-47)}$$

and therefore for any j,

$$a_{ij}' = \frac{a_{rj}}{a_{rk}} \quad \text{for } i = r \qquad \text{and} \qquad a_{ij}' = a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik} \quad \text{for } i \neq r \qquad \text{(A-48)}$$

Transforming g_j and f

If we treat the table of $B \equiv A_0$, the A_j, and the row of g_j (including $j = 0$) as one matrix, we can use the transformation of Equation (A-48) as follows:

1. Divide row r by the pivotal elements a_{rk} giving a new row r.
2. Replace each remaining a_{ij} of the matrix, including the g_j row and including the column $j = 0$ by $a_{ij}' = a_{ij} - a_{rj}'a_{ik}$. A_k' then becomes I_r, g_k' becomes 0, basis vectors other than A_r remain unaltered.
3. Replace c_r by c_k and A_r by A_k in the basis identification columns to the left of the matrix.

The results of these computations can be seen in Iteration 2 of the table titled "Simplex Solution" on page 328. The presence of negative numbers in the Base Row of Iterations 2 and 3 indicate that further improvement is possible, and the process of developing a new basis is repeated. In Iteration 4 all g_j are now $\geqq 0$, indicating that the maximum, or best, solution has been reached for the conditions and restrictions of the problem.

7. Calculating subsequent iterations

After any iteration is completed there is no need to rearrange columns in order to bring the identity submatrix to the left side of coefficients matrix

A. The entire theory developed thus far holds for the columns as they stand by identification of I_i with the appropriate A_j. Therefore, further iterations can be carried by application of the given rules to successive iterations without rearranging columns.

8. Handling degeneracy

Whenever the smallest q_i occurs for several i, then more than one x_i term can become zero. This condition is termed "degeneracy." The function f will show no increase at this stage in the computations which may result in cycling. *Cycling* is the process of introducing and eliminating vectors until the calculations arrive at a previous basis from which point the cycle repeats indefinitely in the same way, getting nowhere.

There is a method for handling situations in which the computational process degenerates or breaks down. Suppose at any stage the sequence of matrix columns is $P_1, \ldots P_n$. The unit basis $A_1, \ldots A_m$ no longer coincides with $P_1, \ldots P_m$, but rather is distributed irregularly throughout the matrix so that $A_j = I_j = P_i$ for $j = 1, \ldots m$ but generally with $j \neq i$ and $1 \leqq i \leqq n$.

Letting ϵ equal a small positive number, consider B replaced by $B' = B + \sum_{J=1}^{n} \epsilon^j P_j$, where ϵ^j is the jth power of ϵ. This modified version will include the original problem by the specialization $\epsilon \rightarrow 0$. Using Equation (A-21) for the P_j, we shall replace $A_1x_1 + A_2x_2 + \cdots + A_mx_m = B$ by

$$\begin{aligned} A_1(x_1 + \epsilon p_{11} + \epsilon^2 p_{12} + \cdots + \epsilon^n p_{1n}) \\ + A_2(x_r + \epsilon p_{21} + \epsilon^2 p_{22} + \cdots \epsilon^n p_{2n}) \\ + \cdots + A_m(x_m + \epsilon p_{m1} + \epsilon^2 p_{m2} + \cdots + \epsilon^n p_{mn}) \\ = B + \epsilon P_1 + \epsilon^2 P_2 + \cdots + \epsilon^n P_n \end{aligned} \tag{A-49}$$

Using $\xi = x_i + \epsilon p_{i1} + \epsilon^2 p_{i2} + \cdots + \epsilon^n p_{in}$ and substituting in Equation (4-4), we obtain the following:

$$\begin{aligned} f &= c_1\xi_1 + c_2\xi_2 + \cdots c_m\xi_m \\ &= (c_1x_1 + \cdots c_mx_m) + \epsilon f_1 + \epsilon^2 f_2 + \cdots \epsilon^n f_n \end{aligned} \tag{A-50}$$

As in the derivation of Equation (A-29) the addition of the product of θ and the identity $A_k - (A_1a_{1k} + \cdots + A_ma_{mk}) = 0$ to Equation (A-49) gives the following expression:

$$A_k \cdot \theta + A_1(\xi_1 - \theta a_{1k}) + \cdots + A_r(\xi_r - \theta a_{rk}) + \cdots + A_m(\xi_m - \theta a_{mk}) = B' \tag{A-51}$$

The subtraction of the product of θ and Equation (A-31) from Equation (A-51) gives the following expression:

$$f' = f - \theta(f_k - c_k) = c_k\theta + c_1(\xi_1 - \theta a_{1k}) + \cdots + c_m(\xi_m - \theta a_{mk}) \quad \text{(A-52)}$$

Suppose that in choosing r there is a tie for the i = values, $i_1 < i_2 < \cdots i_r < \cdots$ are applied to Equation (A-51).

For those i which gave

$$\frac{x_{ij}}{a_{i_1k}} = \frac{x_{i2}}{a_{i_2k}} = \cdots$$

we now compare

$$\begin{aligned} q_{i_1} &= \frac{\xi_{i_1}}{a_{i_1k}} = \frac{x_{i_1}}{a_{i_2k}} + \epsilon\frac{p_{i_11}}{a_{i_1k}} + \epsilon^2\frac{p_{i_12}}{a_{i_1k}} + \cdots \epsilon^n\frac{p_{i_1n}}{a_{i_1k}} \\ q_{i_2} &= \frac{\xi_{i_2}}{a_{i_2k}} = \frac{x_{i_2}}{a_{i_2k}} + \epsilon\frac{p_{i_21}}{a_{i_2k}} + \epsilon^2\frac{p_{i_22}}{a_{i_2k}} + \cdots \epsilon^n\frac{p_{i_2n}}{a_{i_2k}} \end{aligned} \quad \text{(A-53)}$$

These ratios involve the tied rows and the columns $P_1, P_2, \ldots P_n$ in this order for the powers $\epsilon, \epsilon^2, \ldots \epsilon^n$. For a sufficiently small ϵ, P_1 will make some of the factors in Equation (A-52) larger than others and, thereby, eliminate the corresponding rows from further competition involving higher powers of ϵ.

For the remaining tied rows, P_2 will then make some of the factors of ϵ^2 larger than others and in so doing eliminate corresponding rows from further competition. The same holds true for $P_3, P_4, \ldots P_n$.

At no time can all the remaining (more than one) tied rows be eliminated from competition. On reaching comparison for factors of ϵ^j (whenever P_j is one of the unit columns with its nonzero element in one of the tied rows) for example $P_j = I_{i_v}$ where $p_{ij} = 1$ for $i = i_v$ and $p_{ij} = 0$ for $i \neq i_v$ will eliminate i_v if it has not already been eliminated. Because all I_{i_v} (for all tied rows i) are present among the P_j, the process will break all ties. As a result there will be a uniquely determined smallest q_i giving the following:

$$\theta = \frac{\min}{v = 1, 2, \ldots} \frac{1}{a_{i_v}k}(x_{i_v} + \epsilon p_{i_v1} + \epsilon^2 p_{i_v2} + \cdots \epsilon^n P_{i_vn}) \quad \text{(A-54)}$$

Among the coefficients $\xi_{i_v} - \theta a_{i_vk}$, all of which had previously been equal to zero, there will now be one smallest equal to zero with all others being greater than zero. In Equation (A-51) only one of the basis vectors A_{i_v} will be eliminated and degeneracy is avoided.

In the event that ties are encountered again in a later iteration, they can be broken by repeating the same process, taking $P_1, \ldots P_n$ in identical order from left to right. No difficulties will be encountered with this procedure if the original unit basis $A_1, \ldots A_m$ is written at the left-hand side of the coefficients matrix A through which the sequence of P_j will begin its progression in the event of degeneracy.

The preceding analysis can be summarized into a rule as follows:

In the case of ties for r, consider P_t and A_k starting with $t = 1$. For those i which are tied form quotients $q_i = p_{it}/a_{ik}$ and take r equal to the i corresponding to the algebraically smallest q_i (negative p_{it} may make q_i negative). If ties remain consider i for the remaining ties. Increase t by 1 and repeat. After r is determined proceed with the transformation.

The presence of m unit vectors guarantees that any ties must be broken before t reaches the value n

Simplex Solution
Determining Product Mix of Highest Profit Margin

			0	0	0	$-M$	6	4	3	
		P_0	P_4	P_5	P_6	P_7	P_1	P_2	P_3	
0	P_4	48	1	0	0	0	2	4	3	
0	P_5	60	0	1	0	0	4	2	3	
0	P_6	36	0	0	1	0	3	0	1	Iteration 1
← $-M$	P_7	5	0	0	0	1	0	1	0	
		$-5M$	0	0	0	0	-6	$-M-4$	-3	
0	P_4	28	1	0	0	0	2	0	3	
0	P_5	50	0	1	0	0	4	0	3	
← 0	P_6	36	0	0	1	0	3	0	1	Iteration 2
→ 4	P_2	5	0	0	0	1	0	1	0	
		20	0	0	0	$+M+4$	-6	0	-3	
0	P_4	4	1	0	$-2/3$	0	0	0	$7/3$	
← 0	P_5	2	0	1	$-4/3$	0	0	0	$5/3$	
→ 6	P_1	12	0	0	$1/3$	0	1	0	$1/3$	Iteration 3
4	P_2	5	0	0	0	1	0	1	0	
		92	0	0	$+2$	$+M+4$	0	0	-1	
0	P_4	1.2	1	$-7/5$	$+6/5$	0	0	0	0	
→ 3	P_3	1 2	0	$3/5$	$-4/5$	0	0	0	1	
6	P_1	11.6	0	$-1/5$	$3/5$	0	1	0	0	Iteration 4
4	P_2	5	0	0	0	1	0	1	0	
		93.20	0	+0 6	*+1 2	$+M+4$	0	0	0	

9. Convergence

At each iteration, f increases progressively so that the same solution cannot appear at a later iteration. Further, the n-column vectors admit at most $\binom{n}{m}$ different bases, and any solution for any one basis must be unique Therefore there can exist at most $\binom{n}{m}$ different basic feasible solutions, and since none can reappear the procedure will terminate in a finite number of steps. Although no previous solution can appear, it does happen that individual vectors are removed from the basis and later are brought back in or that a vector is introduced and then removed later. It is not the presence of any particular basis vector but the total number of basis vectors that is nonrepetitive. Experience indicates that the best or maximal solution is reached generally in m to $2m$ iterations

10. Duality

Whenever a linear programming problem is solved for a maximum answer by the simplex method, there exists a specific minimum answer involving the same data as the original problem. As a matter of fact, a finite maximum solution exists only if a specific solution to the minimum problem exists. This relationship is termed symmetric, and one problem is said to be the dual of the other.

In matrix form the maximum and minimum problems can be set forth as follows:

Maximum

$$f = \Sigma c_j x_j = \text{maximum} \tag{A-5}$$

where

$$A_1x_1 + A_2x_2 + \cdots A_nx_n \leqq B \tag{A-6}$$

$$x_j \geqq 0 \text{ for all values of } j \tag{A-2}$$

Minimum

$$\phi = \Sigma d_j y_j = \text{minimum} \tag{A-55}$$

where

$$A_1y_1 + A_2y_2 + \cdots A_{\bar{n}}y_{\bar{n}} \geqq C \tag{A-56}$$

$$y_j \geq 0 \text{ for all values of } j \tag{A-57}$$

It is possible to demonstrate and prove mathematically that finite solutions exist either for both problems or for neither. The solution of (A-5) simultaneously provides the solution for (A-55).

Whenever most inequalities are of the form

$$\sum_{j=1}^{\bar{n}} a_{ij}x_j \geqq b_i \tag{A-8}$$

or whenever the number of restrictions greatly exceeds the number of variables, the matrices involved in the calculations may be materially reduced by replacing such a problem by its dual, possibly making use of

$$f = -\phi = \Sigma(-d_j)y_j = \Sigma c_j x_j \quad \text{maximum}$$

APPENDIX B

Comparison of the Simplex Method and the Modi Method

This section compares the simplex method with the modi method, which is a specialized form of the transportation method.

A mathematical discussion and analysis of the relationship of the simplex method to the *general* transportation method may be found in the appendix to "The Stepping Stone Method of Explaining Linear Programming Calculations in Transportation Problems," by A. Charnes and W. W. Cooper. [1]

The simplex method is the basic computational method of LP. The modi method is a special transportation-type method derived from the simplex method.

The modi can be used only if the following conditions are met:

1. All pertinent problem information must be expressed in the same units.
2. The outputs must equal inputs (amounts available must equal amounts demanded).

Differences between the two computational methods can be seen by solving the same problem by both methods and then comparing the arrangement of the problem information as set up for computation, the various programs, and the final solutions.

1. The problem

As a part of a highway-construction program it is necessary to fill in certain low spots to level out the roadbed. It is also necessary in some spots to "cut off" some of the high spots to obtain a level roadbed. The problem therefore is to fill in low spots, each of which requires a specific number of standard truckloads of dirt taken from the high spots. In total, the high spots, or cuts, have a supply of truckloads of dirt equal to that required by the fills. Assuming that all trucks haul the same load per trip and the same number of tons per trip and that the operating cost per mile is the same for each truck, the problem is to determine how the fills can be filled in from the cuts with the least number of truckload miles traveled (minimum cost).

2. Problem information

Mileage Chart

From	To	Total miles
A	I	1
A	II	2
A	III	3
B	I	2
B	II	1
B	III	3
C	I	1
C	II	2
C	III	4

Supply of Dirt

Cut	Truckloads available
A	160
B	50
C	90
Total	300

Demand for Dirt

Fill	Truckloads required
I	100
II	140
III	60
Total	300

3. Setup of the problem information for modi solution

Program 1

		I	II	III	
	C \ R	−1	−2	−4	Truckloads available
A	0	−1 100	2− 60	✓ −3 −4	160
B	1	−2 0	−1 50	−3 -3	50
C	0	−1 −1	−2 30	−4 60	✓ 90
Truckloads required		100	140	60	300

	I	II	III
A	P_1	P_2	P_3
B	P_4	P_5	P_6
C	P_7	P_8	P_9

Number of Truckload Miles

$100 \times 1 = 100$
$60 \times 2 = 120$
$50 \times 1 = 50$
$30 \times 2 = 60$
$60 \times 4 = 240$

Total 570

Program 1 is the first modi program. The requirement at each fill has been met and the capacity of each cut has not been exceeded. It is a feasible program.

Program 2

		I	II	III	Truckloads available
	C \ *R*	−1	−2	−3	
A	0	−1 / 100	−2 / 0	−3 / 60	160
B	1	−2 / 0	−1 / 50	−3 / −2	50
C	0	−1 / −1 alt.	−2 / 90	−4 / −3	90
Truckloads required		100	140	60	300

Program 2 is the second and also the best program. The requirement at each fill has been met, and the capacity of each cut has not been exceeded.

An alternate best program can be obtained by assigning part of the capacity of C to I—indicated by "alt."

Number of Truckload Miles

100 × 1 =	100
60 × 3 =	180
50 × 1 =	50
90 × 2 =	180
Total	510

Alternate Best Program

		I	II	III	Truckloads available
	C \ *R*	−1	−2	−3	
A	0	−1 / 10	−2 / 90	−3 / 60	160
B	1	−2 / 0	−1 / 50	−3 / −2	50
C	0	−1 / 90	−2 / −2	−4 / −3	90
Truckloads required		100	140	60	300

The alternate best program provides a different way of meeting the requirements but at the same number of truckload miles as Program 2.

Number of Truckload Miles

10 × 1 =	10
90 × 2 =	180
60 × 3 =	180
50 × 1 =	50
90 × 1 =	90
Total	510

4. Setup of problem information for simplex solution

Functional or objective equation:

$$\text{Minimum truckload miles} = -1(1P_1 + 2P_2 + 3P_3 + 2P_4 + 1P_5 + 3P_6 + 1P_7 + 2P_8 + 4P_9 + 0P_{10} + 0P_{11} + 0P_{12} + MP_{13} + MP_{14} + MP_{15})$$

Capacity equations:

A $160 = P_1 + P_2 + P_3 + P_{10}$
B $50 = P_4 + P_5 + P_6 + P_{11}$
C $90 = P_7 + P_8 + P_9 + P_{12}$

Requirements equations:

	I	II	III	
A	P_1	P_2	P_3	160
B	P_4	P_5	P_6	50
C	P_7	P_8	P_9	90
	100	140	60	

$\text{I} = 100 = P_1 + P_4 + P_7 + P_{13}$
$\text{II} = 140 = P_2 + P_5 + P_8 + P_{14}$
$\text{III} = 60 = P_3 + P_6 + P_9 + P_{15}$

See the simplex matrix on page 336 for setup of the problem.

5. Comparison of final programs and matrices of each method

A study of values in the final program under each method of solution permits the following comparisons:

1. The value of any vacant square P_n in the final modi matrix, which is obtained by comparing $R + C$ with the corresponding subsquare value is identical to the final simplex matrix Base Row value for the same P_n.

	Modi vacant square values	Simplex base row values
P_2	$0 - 2 = -2 - (-2) = -2 + 2 = 0$	0
P_4	$1 - 1 = 0 - (-2) = 0 + 2 = 2$	2
P_6	$1 - 3 = -2 - (-3) = -2 + 3 = 1$	1
P_9	$0 - 3 = -3 - (-4) = -3 + 4 = 1$	1

2. The path followed in filling a vacant square P_n in the modi will have numerical entries in the corresponding P_n Column in the simplex matrix. The P designations in the Stub corresponding to these entries in the P_n Column are the same as the P designations of the squares in the modi path. The "load" value for each P in the stub is identical to the "load" value for the corresponding P in the modi matrix.

To fill P_6 in the modi, it is necessary to trace a path from $+P_5 - P_2 + P_3$. In the simplex the P_6 Column, neglecting the Base Row, shows entries of -1, 1, and 1. The entries in the Stub which correspond to the entries in the P_6 Column are P_2, P_5, and P_3, which are the same as the modi. The corresponding circled values in the modi path, 50, 0, and 60, are the same as those in the Stub that have entries in the P_6 Column. They are 0, 50, and 60. This relationship holds for the other vacant squares in the modi.

3. For each unit of P_4 brought into solution after the best solution has been calculated, there will result a \$2 loss $(0 - 2 = -2)$. By multiplying the entries in Column P_4 in final solution of simplex by the Stub value and subtracting the Cap value, the same value of \$2 is obtained. This relationship holds for P_6 and P_9 in the modi and the simplex.

	Modi vacant square	Simplex base row
P_6	$-2 - (-3) = 1$	1
P_9	$-3 - (-4) = 1$	1

It does not hold for P_7 because P_7 is an alternate and a loss does not result by loading the square.

4. Values in subsquares of the modi are the Cap values of the simplex without the slack variables. Under the conditions the modi Cap values and entries correspond to the functional equation of the simplex, and each modi square corresponds to a simplex column.

5. Modi has no slack variables. The process of loading eliminates all slack or unused capacity because of the equality of supply and demand.

6. An alternate best answer is indicated in the modi when the sum of the R and C values equals the subsquare value (the difference is zero). In the simplex, an alternate best answer is indicated by a zero in the Base Row except for slack variable columns, columns having entries in the Stub, and columns that were in solution in previous programs. P_7 is the only alternate under modi or simplex that meets the necessary requirements.

Simplex Matrix

			0	0	0	$-M$	$-M$	$-M$	-1	-2	-3	-2	-1	-3	-1	-2	-4
		P_0	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
0	P_{10}	160	1						1	1	1						
0	P_{11}	50		1								1	1	1			
0	P_{12}	90			1										1	1	1
← $-M$	P_{13}	100				1			1			1			1		
$-M$	P_{14}	140					1			1			1			1	
$-M$	P_{15}	60						1			1			1			1
		$-300M$	0	0	0	0	0	0	$-M+1$	$-M+2$	$-M+3$	$-M+2$	$-M+1$	$-M+3$	$-M+1$	$-M+2$	$-M+4$
0	P_{10}	60	1							1	1	-1			-1		
← 0	P_{11}	50		1								1	1	1			
0	P_{12}	90			1										1	1	1
→ -1	P_1	100				1			1			1			1		
$-M$	P_{14}	140					1			1			1			1	
$-M$	P_{15}	60						1			1			1			1
		-100 $-200M$	0	0	0	$M-1$	0	0	0	$-M+2$	$-M+3$	1	$-M+1$	$-M+3$	0	$-M+2$	$-M+4$

Program 1
No dirt moved
No truckload miles

Program 2
100 truckloads moved from Cut A to Fill I
100 truckload miles

← 0	P_{10}	60	1							1	1	−1			−1			
→ −1	P_5	50	0	1								1	1	1				
0	P_{12}	90			1										1	1	1	Program 3
−1	P_1	100				1			1			1			1			100 truckloads moved from Cut A to Fill I
$-M$	P_{14}	90		−1			1			1		−1		−1		1		50 truckloads moved from Cut B to Fill II
$-M$	P_{15}	60						1			1			1			1	100 truckload miles
		-150 $-150M$	0	$M-1$	0	$M-1$	0	0	0	$-M+2$	$-M+3$	M	0	2	0	$-M+2$	$-M+4$	
→ −2	P_2	60	1							1	1	−1			−1			
−1	P_5	50		1								1	1	1				Program 4
0	P_{12}	90			1										1	1	1	100 truckloads moved from Cut A to Fill I
−1	P_1	100				1			1			1			1			50 truckloads moved from Cut B to Fill II
← $-M$	P_{14}	30	−1	−1			1				−1			−1	1	1		60 truckloads moved from Cut A to Fill II
$-M$	P_{15}	60						1			1			1			1	270 truckload miles
		-170 $-90M$	$M-2$	$M-1$	0	$M-1$	0	0	0	0	1	2	0	2	$-M+2$	$-M+2$	$-M+4$	
−2	P_2	60	1							1	1	−1			−1			
−1	P_5	50		1								1	1	1				Program 5
0	P_{12}	60		1	1		−1				1			1			1	100 truckloads moved from Cut A to Fill I
−1	P_1	100				1			1			1			1			50 truckloads moved from Cut B to Fill II
→ −2	P_8	30	−1	−1			1				−1			−1	1	1		60 truckloads moved from Cut A to Fill II
← $-M$	P_{15}	60						1			1			1			1	30 truckloads moved from Cut C to Fill II
		-330 $-60M$	0	1	0	$M-1$	$M-2$	0	0	0	$-M+3$	2	0	$-M+4$	0	0	$-M+4$	330 truckload miles

			0	0	0	$-M$	$-M$	$-M$	-1	-2	-3	-2	-1	-3	-1	-2	-4
		P_0	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
-2	P_2	0	1					-1				-1		-1	-1		-1
-1	P_5	50		1								1	1	1			
0	P_{12}	0		1	1		-1	1									
-1	P_1	100				1			1			1			1		
← -2	P_8	90	-1	-1			1	1		1					1	1	1
→ -3	P_3	60						1			1			1			1
		-510	0	1	0	$M-1$	$M-2$	$M-3$	0	0	0	2	0	1	0	0	1

Program 6

100 truckloads moved from Cut A to Fill I
50 truckloads moved from Cut B to Fill II
90 truckloads moved from Cut C to Fill II
60 truckloads moved from Cut A to Fill III
510 truckload miles

Alternate Best Program

		P_0	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
-2	P_2	90		-1			1			1		-1		-1	0	1	
-1	P_5	50		1								1	1	1	0	1	
0	P_{12}	0		1	1		-1	1							0		
-1	P_1	10	1	1		1	-1	-1	1	-1		1			0	-1	-1
→ -1	P_7	90	-1	-1			1	1		1					1	1	1
-3	P_3	60						1			1			1	0		1
		-510	0	1	0	$M-1$	$M-2$	$M-3$	0	0	0	2	0	1	0	0	1

90 truckloads moved from Cut A to Fill II
50 truckloads moved from Cut B to Fill II
10 truckloads moved from Cut A to Fill I
90 truckloads moved from Cut C to Fill I
60 truckloads moved from Cut A to Fill III
510 truckload miles

Comparison of the programs under both methods shows that the best answer is obtained in two "iterations" by the modi method and six iterations by the simplex. Both methods produce the same final program of allocation of dirt from cuts to fills and the same number of truckload miles traveled.

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